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Bifurcation and Pattern Formation of a Susceptible-Infected-Recovered Model with Diffusion

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Abstract: This paper investigates a susceptible-infected-recovered (SIR) epidemic model with diffusion. The model incorporates psychological impact coefficients a and b, and employs a saturated incidence rate to characterize the psychological effects induced by the disease. First, a priori estimates for positive solutions are derived, and the nonexistence of nonconstant positive solutions is proved by the energy method. Second, the Turing instability of the positive equilibrium is analyzed, and the existence of nonconstant positive solutions is established by means of Leray-Schauder degree theory. Subsequently, bifurcation theory is utilized to characterize the local and global structures of steady state bifurcations. Through spatial decomposition and the implicit function theorem, the local structure of steady state bifurcations at a double eigenvalue is constructed, and a criterion for judging the direction of bifurcation is derived. Finally, numerical simulations are performed to supplement and verify the theoretical results.

Key words: bifurcation; pattern formation; susceptible-infected-recovered (SIR) model; numerical simulation

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0 Introduction

Infectious diseases endanger human health, and exploring their transmission mechanisms is vital for formulating prevention and control strategies. As a core tool for epidemic research, the classical susceptible-infected-recovered (SIR) model by Kermack and McKendrick[1] classifies populations into susceptible (S), infected (I) and recovered (R) groups and describes their transitions, establishing the basis of infectious disease dynamics.

With advances in epidemiological modeling, more realistic biological mechanisms have been integrated into compartmental models. Turing[2] put forward the reaction-diffusion framework in 1952, revealing that diffusion can destabilize homogeneous equilibria and produce spatial patterns, which underpins spatial-pattern research in epidemiology. Ma *et al*[3] established qualitative theory and stability analysis tools for ordinary differential equations for nonlinear dynamica-system studies. Fitzgibbon *et al*[4] constructed a reaction-diffusion model accounting for both direct human-to-human and indirect environment-mediated disease transmission. Brauer *et al*[5] presented a comprehensive framework for population epidemiological modeling.

Building on these advances, researchers have explored the spatiotemporal dynamics of epidemic models. Sun *et*

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al[6] derived Turing instability conditions for a diffusive epidemic model to illustrate how infection and recovery rates shape spatial patterns. Xue *et al*[7] obtained Hopf bifurcation conditions at the endemic equilibrium for a delayed SIR model with logistic growth. Deng *et al*[8] studied SIS reaction-diffusion epidemic dynamics, whereas Cui *et al*[9] examined SIS models under heterogeneous advection. Li and Tong *et al*[10-11] further analyzed diffusive SIS models with variable total populations and logistic sources. These works show that diffusion, population growth and spatial heterogeneity strongly affect disease persistence and spatial spread.

More recent work has integrated complex spatial structures, human behavioral responses and intervention mechanisms into epidemic models. Chang *et al*[12] explored cross-diffusion-driven patterns for network-based SIR models and clarified network-pattern correlations, and further proposed sparse optimal control strategies for SIR reaction-diffusion systems[13]. Goel *et al*[14] constructed a susceptible-aware-infected-recovered (SAIR) model with awareness effects, nonlinear incidence, saturated treatment and time delay. Li *et al*[15] analyzed spatial dynamics of a diffusive SIRS model featuring general saturation and infected-to-susceptible transitions. Collectively, these efforts reflect an evolution from classical temporal compartmental models toward realistic spatial reaction-diffusion frameworks.

To capture environmental factors, population dynamics, disease-induced mortality and nonlinear transmission, Ma *et al*[3] proposed the following epidemic model

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = rS\left(1-\dfrac{S}{K}\right) - \dfrac{\beta SI}{S+I} + cI - \mu S, & t>0, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta SI}{S+I} - \tau I - cI - \mu I, & t>0, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \mu I - \tau R, & t>0. \end{cases} \tag{1}$$

Here, S, I and R represent the densities of susceptible, infected and recovered individuals, respectively, t represents time, K is the environmental carrying capacity, r is the intrinsic growth rate, μ represents the natural death rate of the population, τ is the mortality rate due to the disease, c is the recovery rate of diseased individuals, and β is the transmission coefficient of the infectious disease.

Despite these advances, certain limitations still exist. Traditional ordinary differential equation (ODE) epidemic models assume well-mixed populations and cannot capture spatial heterogeneity caused by human movement. Although reaction-diffusion models overcome this drawback, most existing studies focus on diffusion, time delay, population growth, heterogeneous environments and network structures[6-15]. Few works systematically explore the coupled effects of spatial diffusion and media-induced public behavioral responses. In particular, how such psychological behaviors affect Turing instability, nonconstant positive steady states, and local and global bifurcations in diffusive SIR models remains unclear.

Motivated by this gap, this work incorporates media-driven behavioral responses into a reaction-diffusion SIR framework. We introduce psychological coefficients in the saturated incidence rate to characterize reduced contacts due to public awareness, while diffusion terms describe the random movement of susceptible and infected populations. The proposed model is presented as follows

$$\begin{cases} \dfrac{\partial S}{\partial t} = rS\left(1-\dfrac{S}{K}\right) - \dfrac{\beta SI}{aS+bI} - \mu S + cI + d_1\Delta S, & x\in\Omega, t>0, \\ \dfrac{\partial I}{\partial t} = \dfrac{\beta SI}{aS+bI} - (\mu+\tau+c)I - vI + d_2\Delta I, & x\in\Omega, t>0, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = 0, & x\in\partial\Omega, t>0, \\ S(x,0)=S_0(x)>0, I(x,0)=I_0(x)\geqslant(\not\equiv)0, & x\in\Omega, \end{cases} \tag{2}$$

where Δ is the Laplace operator, n is the outward unit normal vector, Ω is a bounded domain in $\mathbb{R}^N (N\geqslant 1)$ with a smooth boundary $\partial\Omega$, and the homogeneous Neumann boundary condition is being considered. d_1 is the random diffusion coefficient of the susceptible population S, and d_2 denotes the random diffusion coefficient of the infected population I. The diffusion coefficients d_1 and d_2, the initial data $S_0(x)$ and $I_0(x)$, are all positive constants, v denotes the permanent immunity rate, and other environmental factors (e.g., nutrients or contaminants) affect the survival and transmission of infected indi-

viduals. The transmission rate is $\frac{\beta}{aS+bI}$, a and b are constant coefficients representing the psychological influence.

This paper mainly studies the dynamics of positive steady states of system (2), as well as the following steady states problem with the elliptic system

$$\begin{cases} -d_1\Delta S = rS\left(1-\frac{S}{K}\right) - \frac{\beta SI}{aS+bI} - \mu S + cI, & x\in\Omega, \\ -d_2\Delta I = \frac{\beta SI}{aS+bI} - (\mu+\tau+c)I - vI, & x\in\Omega, \\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{3}$$

This paper first investigates the unique positive constant solution to system (3). The a priori estimates for positive solutions are derived by the maximum principle, and the existence of nonconstant positive solutions is proved by combining the energy method with the Leray-Schauder degree theory. Secondly, with parameters r, a, b, μ, K, c, τ and v fixed, we take the diffusion coefficient d_2 as the bifurcation parameter. Local bifurcation theory is adopted to study bifurcation phenomena corresponding to simple and double eigenvalues. As complete bifurcation curves cannot be characterized merely by local bifurcation, an extension to global bifurcations is further performed. The results reveal that when d_2 exceeds a certain critical threshold, the bifurcation curves can extend infinitely with multiple bifurcation points. Finally, numerical simulations are carried out for verification, and the evolutionary mechanism of patterns for uninfected and infected populations with local diffusion is revealed. It is demonstrated that diffusion can drive the system to evolve patch-like, stripe-like and pore-like patterns.

Let $0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_i<\cdots$ represent the eigenvalues of the operator $-\Delta$ under the homogeneous Neumann boundary condition on Ω where each eigenvalue λ_i has a multiplicity $m_i\geqslant 1$. Define $\boldsymbol{\phi}_{ij}(1\leqslant j\leqslant m_i)$ as the normalized eigenfunctions associated with λ_i. Together, $\boldsymbol{\phi}_{ij}, i\geqslant 0,\ 1\leqslant j\leqslant m_i,$ constitute a complete orthogonal basis in $L^2(\Omega)$.

The paper is structured as follows. In Section 1, we establish a priori estimates and derive the nonexistence results for nonconstant positive solutions. In Section 2, we discuss the Turing instability of equilibria and prove the existence of nonconstant positive solutions. In Section 3, we employ bifurcation theorems to analyzed local and global bifurcations, including the bifurcation direction. In Section 4, we supplement and verify the theoretical results by means of numerical simulations.

1 Long-Term Behaviour of the Solutions

In this section, we investigate the persistence properties and derive a priori estimates for the positive steady state of system (3). Furthermore, we prove the nonexistence of nonconstant solutions.

1.1 A Priori Estimates

In this subsection, we will derive a priori estimates for system (3) based on the following lemma.

Lemma 1 (Maximum Principle[16]) Suppose $h\in C(\bar{\Omega}\times\mathbb{R}^1)$,

1) assume that $\iota\in C^2(\Omega)\cap C^1(\bar{\Omega})$, and satisfies $\Delta\iota(x)+h(x,\iota(x))\geqslant 0,\ x\in\Omega,\ \frac{\partial\iota}{\partial n}\leqslant 0,\ x\in\partial\Omega,$ if $\iota(x_0)=\max_{\bar{\Omega}}\iota,$ then $h(x_0,\iota(x_0))\geqslant 0$;

2) assume that $\iota\in C^2(\Omega)\cap C^1(\bar{\Omega})$, and satisfies $\Delta\iota(x)+h(x,\iota(x))\leqslant 0,\ x\in\Omega,\ \frac{\partial\iota}{\partial n}\geqslant 0,\ x\in\partial\Omega,$ if $\iota(x_0)=\min_{\bar{\Omega}}\iota,$ then $h(x_0,\iota(x_0))\leqslant 0$.

Theorem 1 If system (3) admits positive solutions $(S(x),\ I(x))$, and $\mu<r, \beta>aM,\ M=\mu+\tau+c+v$, then

$$0<S(x)<K-\frac{\mu K}{r},\ 0<I(x)<\frac{\beta-aM}{bM}\left(K-\frac{\mu K}{r}\right),\ \forall x\in\bar{\Omega}.$$

Furthermore, if $\beta<r-\mu,$ then $(S(x),\ I(x))$ satisfies

$$K-\frac{(\beta+\mu)K}{r}<S(x)<K-\frac{\mu K}{r},\ \frac{\beta-aM}{bM}\left(K-\frac{(\beta+\mu)K}{r}\right)<I(x)<\frac{\beta-aM}{bM}\left(K-\frac{\mu K}{r}\right).$$

Proof By the Lemma 1, from the first equation of system (3), it follows that $S(x)<K-\frac{\mu K}{r}$. To derive the upper bound for $I(x)$, let $I(x_0)=\max_{\bar{\Omega}} I(x)$. According to the Lemma 1 again, we have $\frac{\beta S(x_0)I(x_0)}{aS(x_0)+bI(x_0)} \geqslant(\mu+\tau+c)I+vI$. Thus

$$I(x_0)\leqslant \frac{1}{b}\left(\frac{\beta I(x_0)}{\mu+\tau+c+v}\right)S(x_0)<\frac{K(r-\mu)\left(\beta-a(\mu+\tau+c+v)\right)}{br(\mu+\tau+c+v)},\quad 0<I(x)<\frac{K(r-\mu)\left(\beta-a(\mu+\tau+c+v)\right)}{br(\mu+\tau+c+v)}.$$

Then find the lower bounds for $S(x)$ and $I(x)$. Let $S(y_0)=\min_{\bar{\Omega}} S(x)$, $I(z_0)=\min_{\bar{\Omega}} I(x)$. We have

$$rS(y_0)\left(1-\frac{S(y_0)}{K}\right)-\beta S(y_0)-\mu S(y_0)<rS(y_0)\left(1-\frac{S(y_0)}{K}\right)-\frac{\beta S(y_0)I(y_0)}{aS(y_0)+bI(y_0)}-\mu S(y_0)+cI(y_0)\leqslant 0,$$

by the Lemma 1, hence, $S(x)>K-\frac{(\beta+\mu)K}{r}$. Then, we have

$$\frac{\beta\frac{K(r-\beta-\mu)}{r}I(z_0)}{a\frac{K(r-\beta-\mu)}{r}+bI(z_0)}-(\mu+\tau+c)I(z_0)-vI(z_0)<\frac{\beta S(z_0)I(z_0)}{aS(z_0)r+bI(z_0)}-(\mu+\tau+c)I(z_0)-vI(z_0)\leqslant 0,$$

thus, $I(x)>\frac{\beta-aM}{bM}\left(K-\frac{(\beta+\mu)K}{r}\right)$. The proof is completed.

1.2 Nonexistence

In this subsection, we use the energy method to discuss the nonexistence of nonconstant positive steady state for system (3). It is shown that if the diffusion coefficients d_1 and d_2 are sufficiently large, system (3) admits no non-constant positive steady-state solutions. Also, to simplify the notation, we let $P=(a,r,c,K,\mu,\beta,\tau,v)$ denote the set of parameters.

Theorem 2 Let $d_2^*=\frac{\mu+\tau+c+v}{\lambda_1}$ be a fixed positive constant, there exists a positive constant d_1^*, which depends on P and d_2^*, such that for any $d_1\geqslant d_1^*$ and $d_2>d_2^*$, system (3) has no nonconstant positive steady state solution.

Proof Let $S=S(x)$ and $I=I(x)$ represent a pair of solutions to system (3). Define $\bar{S}=\frac{1}{|\Omega|}\int_\Omega S(x)\mathrm{d}x$, $\bar{I}=\frac{1}{|\Omega|}\int_\Omega I(x)\mathrm{d}x$. By use a priori estimates, multiply the first equation of system (3) by $\frac{S-\bar{S}}{S}$ and integrate over Ω by parts, it may be inferred that $C_1<\frac{\beta}{aS+bI}<C_2$, then we can derive

$$d_1\int_\Omega\frac{\bar{S}}{S^2}|\nabla S|^2\mathrm{d}x=\int_\Omega\left(-\frac{rS}{K}-\frac{\beta I}{aS+bI}+\frac{cI}{S}\right)(S-\bar{S})\mathrm{d}x\leqslant\int_\Omega\frac{r}{K}(S-\bar{S})^2\mathrm{d}x+\int_\Omega C_2\,(S-\bar{S})(I-\bar{I})\mathrm{d}x+\int_\Omega c(S-\bar{S})(I-\bar{I})\mathrm{d}x.$$

Similarly, multiply the second equation of system (3) by $(I-\bar{I})$ and integrate over Ω, we find that $C_3<\frac{\beta S}{aS+bI}<C_4$.

$$d_2\int_\Omega|\nabla I|^2\mathrm{d}x=\int_\Omega\left(\frac{\beta SI}{aS+bI}-(\mu+\tau+c)I-vI\right)(I-\bar{I})\mathrm{d}x\leqslant\int_\Omega C_4\,(S-\bar{S})(I-\bar{I})\mathrm{d}x+\int_\Omega(\mu+\tau+c+v)(I-\bar{I})^2\,\mathrm{d}x.$$

From Theorem 1, we have a large enough positive constant $C_5^{-1}<S(x),\ I(x),\ \bar{S},\ \bar{I}<C_5$.

According to the previously mentioned estimates show that

$$\begin{aligned}\int_\Omega\left\{\frac{d_1}{C_5^3}|\nabla S|^2+d_2|\nabla I|^2\right\}\mathrm{d}x&\leqslant\int_\Omega\frac{r}{K}(S-\bar{S})^2+2\delta|S-\bar{S}||I-\bar{I}|+(\mu+\tau+c+v)(I-\bar{I})^2\mathrm{d}x\\&\leqslant\int_\Omega(S-\bar{S})^2\left(\frac{r}{K}+\frac{\delta}{\varepsilon}\right)+(I-\bar{I})^2(\mu+\tau+c+v+\delta\varepsilon)\mathrm{d}x.\end{aligned}$$

Utilizing the Poincaré inequality, we obtain

$$\int_\Omega\left\{\frac{d_1\lambda_1}{C_5^3}|S-\bar{S}|^2+d_2\lambda_1|I-\bar{I}|^2\right\}\mathrm{d}x\leqslant\int_\Omega(S-\bar{S})^2\left(\frac{r}{K}+\frac{\delta}{\varepsilon}\right)+(I-\bar{I})^2(\mu+\tau+c+v+\delta\varepsilon)\mathrm{d}x.$$

If $d_2\lambda_1>\mu+\tau+c+v$ holds, we can choose a small enough $\varepsilon_0>0$ so that $d_2\lambda_1>\mu+\tau+c+v+\delta\varepsilon_0$. Hence, we determine that $S=\bar{S}$ and $I=\bar{I}$ by producing $d_1^*=\frac{C_5^3}{\lambda_1}\left(\frac{r}{K}+\frac{\delta}{\varepsilon_0}\right)$. The proof is completed.

2 Existence

In this section, we focus on the Turing instability of system (3), and by Leray-Schauder degree theory, we investigate the possibility of the existence of nonconstant positive steady states for system (3) when diffusion coefficients d_1 and d_2 are appropriately selected and other parameters are fixed.

2.1 Turing Instability

In this subsection, we analyzed the stability of the equilibrium of the diffusion system (2) and draw several conclusions regarding Turing instability. Specifically, we investigate the local dynamics of the system by considering the corresponding system of ODE, which takes the following form

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = rS\left(1-\dfrac{S}{K}\right) - \dfrac{\beta SI}{aS+bI} - \mu S + cI, & t>0, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta SI}{aS+bI} - (\mu+\tau+c)I - vI, & t>0. \end{cases} \tag{4}$$

System (4) has a unique constant positive steady state $E^*(S^*, I^*)$. Here,

$$S^* = \frac{K(-\beta\tau-\beta v-\beta\mu+\mu^2 a+\tau^2 a+v^2 a-\mu^2 b+2\mu\tau a+2\mu v a+c\mu a+c\tau a+cva+rb\mu+rb\tau+rbv+rbc-\mu b\tau-\mu bv-\mu bc)}{rb(\mu+\tau+c+v)},$$

$$I^* = [-K(-\beta\tau-\beta v-\beta\mu+\mu^2 a+\tau^2 a+v^2 a-\mu^2 b+2\mu\tau a+2\mu v a+c\mu a+c\tau a+cva+rb\mu+rb\tau+rbv+rbc-\mu b\tau-\mu bv-\mu bc) - (\beta+\mu a+\tau a+va+ca)]/[rb^2(\mu+\tau+c+v)^2],$$

where β satisfies $(\mu+\tau+c+v)a<\beta<\dfrac{[(\mu+\tau+v)a+(r-\mu)b](\mu+\tau+c+v)}{\mu+\tau+v}$.

After that, we concentrate on the stability of the ODE and PDE systems. The Jacobian matrix of system (4) at E^* is given by $\boldsymbol{J}(E^*)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where

$$\begin{aligned} a_{11} = \frac{\partial f}{\partial S}|_{E^*} = &-(\tau^3 a^3+v^3 a^2+\mu^3 a^2-\beta\mu^2 b+c^3 a^2-\beta\mu b\tau-\beta\mu bv-2\beta c\mu a-2\beta c\tau a-2\beta cva-\beta\mu bc+\beta rb\mu+\beta rb\tau+\beta rbv+\beta rbc \\ &+6\mu a^2\tau c+6\mu a^2 v\tau c-2\beta c^2 a+3\mu^2 a^2\tau+3\mu^2 av+3\mu^2 a^2 c+3\mu a^2\tau^2+3\mu a^2 v^2+3\mu a^2 c^2+3\tau^2 a^2 c+3v^2 a^2 c+3\tau a^2 c \\ &+3va^2 c^2-\beta^2\mu-\beta^2\tau-\beta^2 v+\beta^2 c)\frac{1}{b\beta(\mu+\tau+c+v)}, \end{aligned}$$

$$a_{12} = \frac{\partial f}{\partial I}|_{E^*} = -\frac{-c\beta+\mu^2 a+2\mu\tau a+2\mu va+2c\mu a+\tau^2 a+v^2 a+2c\tau a+2cva+c^2 a}{\beta},\ a_{21} = \frac{\partial g}{\partial S}|_{E^*} = \frac{(\mu a+\tau a+va-\beta+ca)^2}{\beta b},$$

$$a_{22} = \frac{\partial g}{\partial I}|_{E^*} = \frac{\mu^2 a+2\mu\tau a+2\mu va+2c\mu a+\tau^2 a+v^2 a+2c\tau a+2cva-\beta\mu-\beta\tau-\beta v-c\beta+c^2 a}{\beta}.$$

Then, $\boldsymbol{J}(E^*)$ has the characteristic equation

$$\rho^2 - \mathrm{Tr}(\boldsymbol{J})\rho + \mathrm{Det}(\boldsymbol{J}) = 0, \tag{5}$$

where ρ denotes the eigenvalue and

$$\mathrm{Tr}(\boldsymbol{J}) = a_{11}+a_{22},\ \ \mathrm{Det}(\boldsymbol{J}) = a_{11}a_{22}-a_{12}a_{21}.$$

According to Zhou *et al*[17], when $\max\{t_1,t_3\}<\beta<\min\{t_2,t_4\}$, the unique constant positive steady state $E^*$of the ODE system (4) is locally asymptotically stable, where

$$\begin{aligned} t_1 = &\frac{(\mu+\tau+v+c)}{2(\mu-c-\tau-v)}\{b(\tau+v+c+r)-2ac-[2b^2(\tau c+vc+vr+\tau r+cr)+b^2(\tau^2+v^2+c^2+r^2)+8\tau\mu(a^2-ab) \\ &+4a^2(\mu^2+v^2+\tau^2)-4ab(\tau c+rc+\mu^2+\tau^2)]^{\frac{1}{2}}\}, \\ t_2 = &\frac{(\mu+\tau+v+c)}{2(\mu-c+\tau+v)}\{b(\tau+v+c+r)-2ac+[2b^2(\tau c+\tau r+vc+vr+cr)+b^2(\tau^2+v^2+c^2+r^2)+8\tau v\mu(a^2-ab) \\ &+4a^2(\mu^2+\tau^2+v^2)-4ab(\tau c+vc+rc+\mu^2+\tau^2)]^{\frac{1}{2}}\}, \\ t_3 = &a(\mu+\tau+v+c), \qquad t_4 = \frac{[(\mu+\tau+v)a+(r-\mu)b](\mu+\tau+v+c)}{\mu+\tau+v}. \end{aligned}$$

Hence, the ODE system (4) remains stable at the equilibrium E^*. As the proof follows standard procedures, it is

omitted here.

Following this, we investigate the stability of the equilibrium E^* for system (2). If we set

$$\tilde{d}_2=\tilde{d}_2(r,K,a,b,\beta,\mu,\tau,c,v,d_1,\Omega)=\min_{i\leqslant i\leqslant i_0}d_{2,i},\ \text{where}\ d_{2,i}=\frac{a_{11}a_{22}-a_{12}a_{21}-a_{22}d_1\lambda_i}{\lambda_i(a_{11}-d_1\lambda_i)}. \tag{6}$$

Naturally, the following describes the local stability of E^*.

Theorem 3 For the PDE system (2), assume that $\max\{t_1,t_3\}<\beta<\min\{t_2,t_4\}$. Then the following statements hold:

(i) if $a_{11}<0$, the equilibrium E^* is locally asymptotically stable;

(ii) if $a_{11}>0$ then

(ii-1) if $d_1\lambda_1<a_{11}$ and $0<d_2<\tilde{d}_2$, the equilibrium E^* is locally asymptotically stable;

(ii-2) if $d_1\lambda_1<a_{11}$ and $d_2>\tilde{d}_2$, the equilibrium E^* is unstable, and Turing instability occurs.

Proof System (2) has the following linearized form at E^*.

$$\begin{pmatrix}S_t\\I_t\end{pmatrix}=\boldsymbol{L}\begin{pmatrix}S\\I\end{pmatrix}:=\begin{pmatrix}d_1\Delta+a_{11}&a_{12}\\a_{21}&d_2\Delta+a_{22}\end{pmatrix}\begin{pmatrix}S\\I\end{pmatrix}.$$

If $(\boldsymbol{\Phi},\boldsymbol{\Psi})$ is an eigenfunction of $\boldsymbol{L}$ associated with the eigenvalue ρ, then we obtain

$$\begin{pmatrix}d_1\Delta+a_{11}&a_{12}\\a_{21}&d_2\Delta+a_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Phi}\\\boldsymbol{\Psi}\end{pmatrix}=\rho\begin{pmatrix}\boldsymbol{\Phi}\\\boldsymbol{\Psi}\end{pmatrix},$$

whose characteristic equation is

$$\rho^2-P_i\rho+Q_i=0,\ i=0,1,2,\cdots, \tag{7}$$

where $P_i=-(d_1+d_2)\lambda_i+a_{11}+a_{22}$, $Q_i=\lambda_i(d_1\lambda_i-a_{11})\left\{d_2-\dfrac{d_1a_{22}\lambda_i+a_{12}a_{21}-a_{11}a_{22}}{\lambda_i(d_1\lambda_i-a_{11})}\right\}$.

We have $P_i<0$ and $Q_i>0$ for all $i\geqslant 0$, which indicates that $\mathrm{Re}\,\rho<0$ for every eigenvalue ρ. Consequently, the equilibrium E^* is locally asymptotically stable.

In addition, we know that $P_i<0$ and let $d_1a_{22}\lambda_i+a_{12}a_{21}-a_{11}a_{22}<0$. For system (2), we primarily analyze the behaviour of Q_i for $i\geqslant 1$, as outlined below.

If $0<d_2<\tilde{d}_2$, then for $1\leqslant i\leqslant i_0$, we have $d_1\lambda_i<a_{11}$ and $d_2<\tilde{d}_2$, which ensures that $Q_i>0$. For $i\geqslant i_0$, since $d_1\lambda_1>a_{11}$, we still obtain $Q_i>0$. Consequently, $Q_i>0$ holds for all $i\geqslant 1$, indicating that E^* is locally asymptotically stable.

Conversely, if $d_2>\tilde{d}_2$, we assume that the minimum in (6) is attained at some $1\leqslant j\leqslant i_0$, leading to $d_2>d_{2,j}$, which implies $Q_i<0$. That is, E^* is unstable. The proof is completed.

2.2 Existence of Nonconstant Positive Steady States

In this subsection, we explore whether there are nonconstant positive solutions to system (3) by investigating the effect of varying the diffusion coefficients d_1 and d_2 with all other parameters fixed. For this purpose, we prove the existence of nonconstant steady state solutions using the Leray-Schauder degree theorem. Define the function space

$$\mathbb{X}=\{(S,I)\in[H^1(\Omega)]^2:\ \frac{\partial S}{\partial n}=\frac{\partial I}{\partial n}=0\ \text{on}\ \partial\Omega\}.$$

Let $E(\lambda_i)$ denote the corresponding eigenspace associated with the eigenvalue λ_i, with an orthonormal basis given by $\{\boldsymbol{\phi}_{ij}:\ j=1,2,\cdots,\dim E(\lambda_i)\}$. Define the subspace $\mathbb{X}_{ij}=\{c\boldsymbol{\phi}_{ij}:\ c\in\mathbb{R}^2\}$. By this, the space $\mathbb{X}$ can be expressed as the direct sum

$$\mathbb{X}=\oplus_{i=0}^{\infty}\mathbb{X}_i,\quad \mathbb{X}_i=\oplus_{j=1}^{\dim E(\lambda_i)}\mathbb{X}_{ij}.$$

To simplify notation, let $\boldsymbol{U}=(S,I)^{\mathrm{T}}$ represent the state vector and $\boldsymbol{F}=(f_1,f_2)^{\mathrm{T}}$ the associated functions, where

$$f_1=rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{aS+bI}+cI-\mu S,\ f_2=\frac{\beta SI}{aS+bI}-\tau I-cI-\mu I-vI.$$

The Jacobian matrix $\boldsymbol{J}$ of $\boldsymbol{F}$ evaluated at the equilibrium point E^* is written as $\boldsymbol{J}=\boldsymbol{F}_U(E^*)$. Accordingly, system (3) can be rewritten in the following form

$$-\Delta\boldsymbol{U}=\boldsymbol{T}^{-1}\boldsymbol{F}(\boldsymbol{U}),\ x\in\Omega,\quad \frac{\partial\boldsymbol{U}}{\partial n}=0,\ x\in\partial\Omega, \tag{8}$$

where $\boldsymbol{T}=\mathrm{diag}(d_1,d_2)$. The E^* is a solution of (8) if and only if it satisfies

$$\hat{f}(d_1,d_2;\boldsymbol{U}):=\boldsymbol{U}-(\boldsymbol{O}-\Delta)^{-1}\{\boldsymbol{T}^{-1}\boldsymbol{F}(\boldsymbol{U})+\boldsymbol{U}\}=0\ \text{in}\ \mathbb{X},$$

where $\boldsymbol{O}$ denotes the identity matrix.

A straightforward calculation yields

$$D_U\hat{f}(d_1,d_2;E^*)=\boldsymbol{O}-(\boldsymbol{O}-\Delta)^{-1}(\boldsymbol{T}^{-1}\boldsymbol{J}+\boldsymbol{O}).$$

For each subspace $\mathbb{X}_i$, ξ is an eigenvalue of $D_U\hat{f}(d_1,d_2;E^*)$ when restricted to $\mathbb{X}_i$ if and only if $\xi(1+\lambda_i)$ is an eigenvalue of the following matrix

$$\boldsymbol{M_i}:=\lambda_i\boldsymbol{O}-\boldsymbol{T}^{-1}\boldsymbol{J}=\begin{pmatrix}\lambda_i-d_1^{-1}a_{11} & -d_1^{-1}a_{12}\\ -d_2^{-1}a_{21} & \lambda_i-d_2^{-1}a_{22}\end{pmatrix}.$$

The determinant and trace of $\boldsymbol{M_i}$ are given by

$$\det(\boldsymbol{M_i})=d_1^{-1}d_2^{-1}[d_1d_2\lambda_i^2+(-d_1a_{22}-d_2a_{11})\lambda_i+(a_{11}a_{22}-a_{12}a_{21})],\ \ \mathrm{tr}(\boldsymbol{M_i})=2\lambda_i-d_1^{-1}a_{11}-d_2^{-1}a_{22}.$$

Define

$$\hat{g}(d_1,d_2;\lambda):=d_1d_2\lambda^2+(-d_1a_{22}-d_2a_{11})\lambda+(a_{11}a_{22}-a_{12}a_{21}).$$

If $-d_1a_{22}-d_2a_{11}>2\sqrt{d_1d_2(a_{11}a_{22}-a_{12}a_{21})}$, then $\hat{g}(d_1,d_2;\lambda)=0$ has two positive real roots

$$\lambda_{\pm}(d_1,d_2)=\frac{d_1a_{22}+d_2a_{11}\pm\sqrt{(d_1a_{22}+d_2a_{11})^2-4d_1d_2(a_{11}a_{22}-a_{12}a_{21})}}{2d_1d_2}.$$

Define the set $A(d_1,\ d_2)$ and spectrum S_p as $A(d_1,d_2)=\{\lambda:\lambda\geqslant 0,\ \lambda_-(d_1,d_2)<\lambda<\lambda_+(d_1,d_2)\}$, $S_p=\{\lambda_0,\lambda_1,\lambda_2,\cdots\}$, and let $m(\lambda_i)$ represent the multiplicity of λ_i, to calculate the index of $\hat{f}(d_1,d_2;\cdot)$ at E^*, we use the following lemma.

Lemma 2[18] Suppose $\hat{g}(d_1,d_2;\lambda_i)\neq 0$ for all $\lambda_i\in S_p$. Then

$$\mathrm{index}(\hat{f}(d_1,d_2;\cdot),\ E^*)=(-1)^{\sigma},$$

where

$$\sigma=\begin{cases}\displaystyle\sum_{\lambda_i\in A\cap S_p}m(\lambda_i), & A\cap S_p\neq\varnothing,\\ 0, & A\cap S_p=\varnothing.\end{cases}$$

If $\hat{g}(d_1,d_2;\lambda_i)>0$ for all $\lambda_i\geqslant 0$, then $\sigma=0$.

By the Lemma 2, in order to calculate the index of $\hat{f}(d_1,d_2;\cdot)$ at E^*, it is necessary to identify the range of λ for which $\hat{g}(d_1,d_2;\lambda)<0$.

Theorem 4 Suppose $\dfrac{a_{11}}{d_1}\in(\lambda_k,\lambda_{k+1})$ for some $k\geqslant 1$, and $\sigma_k=\displaystyle\sum_{i=1}^{k}m(\lambda_i)$ is an odd integer. Then, there exists a sufficiently large constant d^* such that for all $d_2\geqslant d^*$, system (3) admits at least one nonconstant positive steady state solution.

$$\lambda_+(d_1,d_2)\in(\lambda_k,\lambda_{k+1}),\ 0<\lambda_-(d_1,d_2)<\lambda_1,\ \forall d_2\geqslant d_0. \tag{9}$$

By Theorem 2, it follows that there exists $d_*>d_0$ such that system (3) admits no nonconstant positive steady state solution when $d_1=d_*$ and $d_2\geqslant d_*$. Let $d_*>0$ be sufficiently large so that $\dfrac{a_{11}}{d_1}<\lambda_1$. Thus there exists $d^*>d_*$ such that

$$0<\lambda_-(d_1,d_2)<\lambda_+(d_1,d_2)<\lambda_1\ \text{ for all }\ d_2\geqslant d^*. \tag{10}$$

We now demonstrate that for any $d_2\geqslant d^*$, system (3) possesses at least one nonconstant positive solution. For contradiction, let $d_2^*\geqslant d^*$ be such that no nonconstant positive solution exists for system (3). Next, we use the homotopy argument to derive the contradiction. Fix $d_2=d_2^*$, and for $t\in[0,1]$, we define

$$\boldsymbol{T}(t)=\begin{pmatrix}td_1+(1-t)d_* & 0\\ 0 & td_2+(1-t)d^*\end{pmatrix},$$

and examine the following problem

$$-\Delta\boldsymbol{U}=\boldsymbol{T}^{-1}(t)\boldsymbol{F}(\boldsymbol{U}),\ x\in\Omega,\quad\frac{\partial\boldsymbol{U}}{\partial n}=0,\ x\in\partial\Omega. \tag{11}$$

It is clear that $\boldsymbol{U}$ represents a nonconstant positive steady state of system (3) if and only if it satisfies the equation (11) with $t=1$, Clearly, E^* is the unique positive constant solution of (11). Furthermore, for any $t\in[0,1]$, $\boldsymbol{U}$ is a nonconstant positive steady state of the equation (11) if and only if it fulfills the following functional equation

$$h(\boldsymbol{U};t)=\boldsymbol{U}-(\boldsymbol{O}-\Delta)^{-1}\{\boldsymbol{T}^{-1}(t)\boldsymbol{F}(\boldsymbol{U})+\boldsymbol{U}\}=0\ \text{on}\ \mathbb{X}. \tag{12}$$

Here, there is no nonconstant positive steady state to (12) when $t=0$, and we propose the absence of such a solution for $t=1$ at $d_2=d_2^*$. Denote $h(\boldsymbol{U};1)=\hat{f}(d_1,d_2;\boldsymbol{U})$.

Based on (9) and (10), and when $t=0$, we set $\tilde{\boldsymbol{T}}=\mathrm{diag}(d_*, d^*)$, we know $A(d_1, d_2)\cap S_p=\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ and $A(d_*, d^*)\cap S_p=\varnothing$. Since σ_k is odd, the Lemma 2 implies

$$\mathrm{index}(h(\cdot;1), E^*)=\mathrm{index}(\hat{f}(d_1, d_2;\cdot), E^*)=(-1)^{\sigma_k}=-1,\ \mathrm{index}(h(\cdot;0), E^*)=\mathrm{index}(\hat{f}(d_*, d^*;\cdot), E^*)=(-1)^0=1.$$

By Theorem 1, there exist positive constants $\underline{C}$ and $\bar{C}$ ensuring that the positive solutions of the equation (12) satisfy $\underline{C}<S(x), I(x)<\bar{C}$ for $t\in[0, 1]$.

Let $\Sigma=\{(S, I)^{\mathrm{T}}\in\mathbb{X}: \underline{C}<S(x), I(x)<\bar{C}, x\in\bar{\Omega}\}$. Since $h(\boldsymbol{U}; t)\neq 0$ for $\boldsymbol{U}\in\partial\Sigma$ and $t\in[0, 1]$, by homotopy invariance of the Leray-Schauder degree, we can conclude that

$$\deg(h(\cdot;0), \Sigma, 0)=\deg(h(\cdot;1), \Sigma, 0). \tag{13}$$

However, given $h(\boldsymbol{U}; 0)=0$ and $h(\boldsymbol{U}; 1)=1$ there is a unique positive solution E^* in Σ, we conclude that

$$\deg(h(\cdot;0), \Sigma, 0)=\mathrm{index}(h(\cdot;0), E^*)=1,\ \deg(h(\cdot;1), \Sigma, 0)=\mathrm{index}(h(\cdot;1), E^*)=-1,$$

which contradicts (13). The proof is completed.

3 Bifurcation

This section investigates the structure of the nonconstant positive solutions of system (3) in three subsections. In subsection 3.1, we study the local bifurcation at simple eigenvalues using the Crandall-Rabinowitz theorem and explore the local bifurcation under double eigenvalues using techniques such as spatial decomposition and the implicit function theorem. In subsection 3.2, we use the global bifurcation theory to extend local bifurcations induced by simple eigenvalues to global bifurcations. It is found that these bifurcation curves may encounter certain bifurcation points and extend to infinity under the condition that the parameter d_2 is larger than certain critical values. In subsection 3.3, we study the direction of the local bifurcation at simple eigenvalues.

3.1 Local Structure

In the one-dimensional space interval $\Omega=(0, \pi)$, we now examine the structure of nonconstant positive stable states for system (3). As a result, system (3) may be expressed as

$$\begin{cases} -d_1 S''=rS\left(1-\dfrac{S}{k}\right)-\dfrac{\beta SI}{aS+bI}-\mu S+cI, & x\in\Omega, \\ -d_2 I''=\dfrac{\beta SI}{aS+bI}-(\mu+\tau+c)I-vI, & x\in\Omega, \\ S_x=I_x=0, & x=0, \pi. \end{cases} \tag{14}$$

Here, we choose d_2 as the bifurcation parameter, and use the transformation $(\hat{S}, \hat{I})=(S-S^*, I-I^*)$ to shift (S, I) to the origin. We then continue to represent $(\hat{S}, \hat{I})$ by (S, I). System (14) is therefore transformed into the following

$$\begin{cases} -d_1 S''=r(S+S^*)\left(1-\dfrac{S+S^*}{k}\right)-\dfrac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-\mu(S+S^*)+c(I+I^*), \\ -d_2 I''=\dfrac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-(\mu+\tau+c)(I+I^*)-(I+I^*)v, \\ S_x=I_x=0. \end{cases} \tag{15}$$

Therefore, we will discuss the local bifurcation associated with system (15). To begin with, we need to determine the existence of a positive steady state to the bifurcation from $(d_2, (0, 0))$.

Let $\boldsymbol{X}=\{(S, I)\in C^2(0, \pi)\times C^2(0, \pi): S'=I'=0, x=0, \pi\}$ and $\boldsymbol{Y}=L^p(0, \pi)\times L^p(0, \pi)$. Define the map $\boldsymbol{F}: \mathbb{R}^+\times\boldsymbol{X}\to\boldsymbol{Y}$ by

$$\boldsymbol{F}(d_2, \boldsymbol{U})=\begin{pmatrix} d_1 S''+r(S+S^*)(1-\dfrac{S+S^*}{k})-\dfrac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-\mu(S+S^*)+c(I+I^*) \\ d_2 I''+\dfrac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-(\mu+\tau+c)(I+I^*)-(I+I^*)v \end{pmatrix},\ \boldsymbol{U}=\begin{pmatrix} S \\ I \end{pmatrix}.$$

The zeros on this map therefore exactly match the solutions of (15), clearly, $\boldsymbol{F}(d_2, (0, 0))=0$. Next, we proceed to assess the Fréchet derivative of $\boldsymbol{F}$ at $(0, 0)$.

$$\boldsymbol{L}(d_2)=\boldsymbol{F}_U\big(d_2,\ (0,0)\big)=\begin{pmatrix} d_1\Delta+a_{11} & a_{12} \\ a_{21} & d_2\Delta+a_{22}\end{pmatrix},$$

where $\Delta=\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\boldsymbol{L}(d_2)$ is given by the foregoing equation.

Throughout this subsection, we assume $d_1\lambda_1<a_{11}$, there exists the largest positive integer i_0 such that $d_1\lambda_i<a_{11}$ for $1\leqslant i\leqslant i_0$. Substituting $\rho=0$ into equation (5), we obtain the following relationship

$$\text{(H): } d_2=d_{2,i}:=\frac{a_{11}a_{22}-a_{12}a_{21}-a_{22}d_1\lambda_i}{\lambda_i(a_{11}-d_1\lambda_i)},\ \text{where } \lambda_i=i^2,\ \ 1\leqslant i\leqslant i_0.$$

Our goal is to establish the existence of nonconstant positive steady states for $\boldsymbol{F}\big(d_2,\ (S,I)\big)=0$ in the vicinity of the trivial solution $\boldsymbol{F}\big(d_2,\ (0,0)\big)=0$ for $1\leqslant i\leqslant i_0$. Note that the values of $d_{2,i}$ may coincide or differ for distinct indices i and j ($i\neq j$). We consider two cases: $d_{2,i}=d_{2,j}$ and $d_{2,i}\neq d_{2,j}$, where $i\neq j$.

Theorem 5 Assume condition H is holds,

(i) if $d_{2,i}\neq d_{2,j}$ whenever $i\neq j$, for all $i,j\in[1,i_0]$, then $\big(d_{2,i},\ (0,0)\big)$ represents a bifurcation point of equation $\boldsymbol{F}\big(d_2,\boldsymbol{U}\big)=0$. Moreover, there exists a curve of nonconstant positive steady states, denoted as $\big(d_2(s),\big(\boldsymbol{S}(s),\boldsymbol{I}(s)\big)\big)$, satisfying $\boldsymbol{F}\big(d_2,\boldsymbol{U}\big)=0$ for sufficiently small $|s|$. Specifically, $d_2(0)=d_{2,i}$, $\big(S(0),I(0)\big)=(0,0)$, $\boldsymbol{S}(s)=s\boldsymbol{\phi}_i+o(s)$, and $\boldsymbol{I}(s)=se_i\boldsymbol{\phi}_i+o(s)$, where $d_2(s)$, $\boldsymbol{S}(s)$ and $\boldsymbol{I}(s)$ are continuously differentiable with respect to s. Here, $\boldsymbol{\phi}_i=\sqrt{\dfrac{2}{\pi}}\cos ix$, and $e_i=\dfrac{d_1\lambda_i-a_{11}}{a_{12}}$;

(ii) suppose there exist distinct indices i and j ($i\neq j$) such that $d_{2,i}=d_{2,j}=\hat{d}_2$. Let

$$e_i=\frac{d_1\lambda_i-a_{11}}{a_{12}},\quad e_i^*=\frac{d_1\lambda_i-a_{11}}{a_{21}},\quad \boldsymbol{\Phi}_i=\begin{pmatrix}1\\ e_i\end{pmatrix}\boldsymbol{\phi}_i, \tag{16}$$

$$\begin{cases} A_1=-\dfrac{r}{K}+\dfrac{ab\beta I^{*2}}{(aS^*+bI^*)^3}-\dfrac{2ab\beta S^*I^*}{(aS^*+bI^*)^3}e_i+\dfrac{ab\beta S^{*2}}{(aS^*+bI^*)^3}e_i^2\neq 0,\\ A_2=-\dfrac{2r}{K}+\dfrac{2ab\beta I^{*2}}{(aS^*+bI^*)^3}-\dfrac{2ab\beta S^*I^*}{(aS^*+bI^*)^3}(e_i+e_j)+\dfrac{2ab\beta S^{*2}}{(aS^*+bI^*)^3}e_ie_j\neq 0,\\ A_3=-\dfrac{r}{K}+\dfrac{ab\beta I^{*2}}{(aS^*+bI^*)^3}-\dfrac{2ab\beta S^*I^*}{(aS^*+bI^*)^3}e_j+\dfrac{ab\beta S^{*2}}{(aS^*+bI^*)^3}e_j^2\neq 0, \end{cases} \tag{17}$$

$$X_2:=\{(S,I)^{\mathrm{T}}\in X:\int_0^{\pi}(S+e_iI)\boldsymbol{\phi}_i\mathrm{d}x=\int_0^{\pi}(S+e_jI)\boldsymbol{\phi}_j\mathrm{d}x=0\}. \tag{18}$$

If $1+e_ie_i^*\neq 0$, $1+e_je_j^*\neq 0$, and $j=2i$ $\big(\text{or } i=2j\big)$, then $\big(\hat{d}_2,\ (0,0)\big)$ serves as a bifurcation point for the equation $\boldsymbol{F}\big(d_2,\boldsymbol{U}\big)=0$. Furthermore, there exists a family of nonconstant steady states expressed as $\big(d_2(\omega),\ s(\omega)(\cos\omega\boldsymbol{\Phi}_i+\sin\omega\boldsymbol{\Phi}_j+\boldsymbol{W}(\omega))\big)$, where $\boldsymbol{F}\big(d_2,\boldsymbol{U}\big)=0$, holds for $|\omega-\omega_0|$ sufficiently small. Here, $\big(d_2(\omega),s(\omega),\boldsymbol{W}(\omega)\big)$ are functions continuously differentiable with respect to ω and satisfy the conditions $d_2(\omega_0)=\hat{d}_2$, $s(\omega_0)=0$, and $\boldsymbol{W}(\omega_0)=0$, here ω_0 is any constant satisfying

$$\cos\omega_0\neq 0\ \text{ and }\ A_2c_2c_4e_j\lambda_j\sin^2\omega_0\neq A_1c_1c_3e_i\lambda_i\cos^2\omega_0, \tag{19}$$

$$\big(\text{resp. }\sin\omega_0\neq 0\ \text{ and }\ A_3c_2c_5e_j\lambda_j\sin^2\omega_0\neq A_2c_1c_6e_i\lambda_i\cos^2\omega_0\big), \tag{20}$$

where

$$c_1=\frac{e_i^*}{1+e_ie_i^*},\ \ c_2=\frac{e_j^*}{1+e_je_j^*},\ \ c_3=\sqrt{\frac{1}{2\pi}}\frac{e_j^*-1-\dfrac{r}{KA_1}}{1+e_je_j^*},$$

$$c_4=\sqrt{\frac{1}{2\pi}}\frac{e_i^*-1-\dfrac{r}{KA_2}}{1+e_ie_i^*},\ \ c_5=\sqrt{\frac{1}{2\pi}}\frac{e_i^*-1-\dfrac{r}{KA_3}}{1+e_ie_i^*},\ \ c_6=\sqrt{\frac{1}{2\pi}}\frac{e_j^*-1-\dfrac{r}{KA_2}}{1+e_je_j^*}.$$

Proof To verify statement (i), we utilize the Crandall-Rabinowitz theorem regarding simple eigenvalues[19]. This theorem guarantees that $\big(d_{2,i},\ (0,0)\big)$ is a bifurcation point if the following conditions are satisfied,

1) the derivatives $\boldsymbol{F}_{d_2}$, $\boldsymbol{F}_U$ and $\boldsymbol{F}_{d_2U}$ are defined and continuous;

2) $\dim\ker \boldsymbol{F}_U(d_{2,i},\,(0,0))=\operatorname{codim} R(\boldsymbol{F}_U(d_{2,i},\,(0,0)))=1$;

3) let $\ker \boldsymbol{F}_U(d_{2,i},\,(0,0))=\operatorname{span}\{\boldsymbol{\Phi}_i\}$; then it follows that $\boldsymbol{F}_{d_2U}(d_{2,i},\,(0,0))\boldsymbol{\Phi}_i\notin R(\boldsymbol{F}_U(d_{2,i},\,(0,0)))$.

When $d_{2,i}\neq d_{2,j}$, the operator

$$\boldsymbol{L}(d_{2,i})=\boldsymbol{F}_U(d_{2,i},\,(0,0))=\begin{pmatrix} d_1\Delta+a_{11} & a_{12}\\ a_{21} & d_{2,i}\Delta+a_{22}\end{pmatrix},\ \Delta=\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

is considered. It is straightforward to check that $\boldsymbol{F}_{d_2}$, $\boldsymbol{F}_U$ and $\boldsymbol{F}_{d_2U}$ are continuous, and $\ker \boldsymbol{L}(d_{2,i})=\operatorname{span}\{\boldsymbol{\Phi}_i\}$, $\boldsymbol{\Phi}_i=\begin{pmatrix}1\\ e_i\end{pmatrix}\boldsymbol{\phi}_i$, where $e_i=\dfrac{d_1\lambda_i-a_{11}}{a_{12}}>0$. Hence, we find that $\dim\ker \boldsymbol{L}(d_{2,i})=1$.

Similarly, we can derive that $\ker \boldsymbol{L}^*(d_{2,i})=\operatorname{span}\{\boldsymbol{\Phi}_i^*\}$, $\boldsymbol{\Phi}_i^*=\begin{pmatrix}1\\ e_i^*\end{pmatrix}\boldsymbol{\phi}_i$, where $e_i^*=\dfrac{d_1\lambda_i-a_{11}}{a_{21}}<0$. Since $R(\boldsymbol{L})=(\ker \boldsymbol{L}^*)^{\perp}$, it follows that $\operatorname{codim} R(\boldsymbol{L}(d_{2,i}))=\dim\ker \boldsymbol{L}^*(d_{2,i})=1$.

Finally, for

$$\boldsymbol{F}_{d_2U}(d_{2,i},(0,0))\boldsymbol{\Phi}_i=\begin{pmatrix}0 & 0\\ 0 & \Delta\end{pmatrix}\boldsymbol{\Phi}_i=\begin{pmatrix}0\\ -\lambda_i e_i\boldsymbol{\phi}_i\end{pmatrix},$$

we calculate

$$\left\langle \boldsymbol{F}_{d_2U}(d_{2,i},\,(0,0))\boldsymbol{\Phi}_i,\ \boldsymbol{\Phi}_i^*\right\rangle=-\lambda_i e_i e_i^*\int_0^{\pi}\boldsymbol{\phi}_i^2\mathrm{d}x=-\lambda_i e_i e_i^*\neq 0, \tag{21}$$

thus, $\boldsymbol{F}_{d_2U}(d_{2,i},\,(0,0))\boldsymbol{\Phi}_i\notin R(\boldsymbol{L}(d_{2,i}))$, completing the proof of statement (i).

(ii) Consider $i\neq j$ such that $d_{2,i}=d_{2,j}=\hat{d}_2$. Clearly, $\boldsymbol{F}(\hat{d}_2,\,(0,0))=0$. Let $\boldsymbol{L}(\hat{d}_2)=\boldsymbol{F}_U(\hat{d}_2,\,(0,0))$, we obtain

$$\ker\boldsymbol{L}(\hat{d}_2)=\operatorname{span}\{\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j\},\quad \ker\boldsymbol{L}^*(\hat{d}_2)=\operatorname{span}\{\boldsymbol{\Phi}_i^*,\boldsymbol{\Phi}_j^*\},\ R(\boldsymbol{L}(\hat{d}_2))=\left\{(S,I)^{\mathrm{T}}\in Y:\int_0^{\pi}(S+e_i^*I)\boldsymbol{\phi}_i\mathrm{d}x=\int_0^{\pi}(S+e_j^*I)\boldsymbol{\phi}_j\mathrm{d}x=0\right\}.$$

This implies $\dim\ker\boldsymbol{L}(\hat{d}_2)=\operatorname{codim} R(\boldsymbol{L}(\hat{d}_2))=2$.

As a result, the Crandall-Rabinowitz theorem[19] cannot be used here. Because condition 2) in statement (i) is not fulfilled, we use spatial decomposition and the implicit function theorem to solve this problem.

To proceed, we decompose the solution space. Let $X_1=\ker\boldsymbol{L}(\hat{d}_2)$ and define a projection operator $\boldsymbol{P}:Y\to X_1$ by

$$\boldsymbol{P}\begin{pmatrix}S\\ I\end{pmatrix}=\frac{1}{1+e_ie_i^*}\left[\int_0^{\pi}(S+e_i^*I)\,\boldsymbol{\phi}_i\mathrm{d}x\right]\boldsymbol{\Phi}_i+\frac{1}{1+e_je_j^*}\left[\int_0^{\pi}(S+e_j^*I)\,\boldsymbol{\phi}_j\mathrm{d}x\right]\boldsymbol{\Phi}_j.$$

Here, $R(\boldsymbol{P})=\operatorname{span}\{\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j\}=X_1\subset Y$, and $\boldsymbol{P}^2=\boldsymbol{P}$, making P a projection onto X_1. We decompose Y into $Y=Y_1\oplus Y_2$, where $Y_1=R(\boldsymbol{P})=X_1$ and $Y_2=\ker(\boldsymbol{P})=R(\boldsymbol{L}(\hat{d}_2))$. Similarly, we decompose X into $X_1\oplus X_2$, where $X_1=\operatorname{span}\{\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j\}$ and X_2 is defined in (18).

The solution $\boldsymbol{F}(d_2,\boldsymbol{U})=0$ is then sought in the form $\boldsymbol{U}=s(\cos\omega\,\boldsymbol{\Phi}_i+\sin\omega\,\boldsymbol{\Phi}_j+\boldsymbol{W})$, $\boldsymbol{W}=(\omega_1,\omega_2)^{\mathrm{T}}\in X_2$, where $s,\omega\in\mathbb{R}$ are parameters.

The mapping $\boldsymbol{F}$: $\mathbb{R}^+\times X\to Y$ can then be reformulated as

$$\boldsymbol{F}(d_2,\boldsymbol{U})=\boldsymbol{L}(d_2)\begin{pmatrix}S\\ I\end{pmatrix}+\begin{pmatrix}F^1(S,I)\\ F^2(S,I)\end{pmatrix},$$

where $-F^1(S,I)-\dfrac{r}{K}S^2=F^2(S,I)$.

It follows that (S,I) solves $\boldsymbol{F}(d_2,\boldsymbol{U})=0$ if and only if it satisfies the above formulation. This leads us to investigate the existence of nonconstant pairs (S,I).

Fix $\omega_0\in\mathbb{R}$, and define the nonlinear mapping $\boldsymbol{K}(d_2,s,\boldsymbol{W};\omega)$: $\mathbb{R}^+\times\mathbb{R}\times X_2\times(\omega_0-\delta,\omega_0+\delta)\to Y$ as

$$\boldsymbol{K}(d_2,s,\boldsymbol{W};\omega)=s^{-1}\boldsymbol{F}(d_2,s(\cos\omega\,\boldsymbol{\Phi}_i+\sin\omega\,\boldsymbol{\Phi}_j+\boldsymbol{W})), \tag{22}$$

which can be expanded as

$$\boldsymbol{K}(d_2,s,\boldsymbol{W};\omega)=\boldsymbol{L}(d_2)(\cos\omega\,\boldsymbol{\Phi}_i+\sin\omega\,\boldsymbol{\Phi}_j+\boldsymbol{W})+s\begin{pmatrix}\tilde{F}^1\\ \tilde{F}^2\end{pmatrix},$$

where $-\tilde{F}^1-\dfrac{r}{K}(\cos\omega\boldsymbol{\phi}_i+\sin\omega\boldsymbol{\phi}_j+\omega_1)^2=\tilde{F}^2$, and

$$\tilde{F}^2=-\frac{ab\beta I^{*2}}{(aS^*+bI^*)^3}(\cos\omega\boldsymbol{\phi}_i+\sin\omega\boldsymbol{\phi}_j+\omega_1)^2-\frac{a^2b\beta S^{*2}}{(aS^*+bI^*)^3}(e_i\cos\omega\boldsymbol{\phi}_i+e_j\sin\omega\boldsymbol{\phi}_j+\omega_2)^2$$
$$+\frac{ab\beta S^*I^*}{(aS^*+bI^*)^3}(\cos\omega\boldsymbol{\phi}_i+\sin\omega\boldsymbol{\phi}_j+\omega_1)(e_i\cos\omega\boldsymbol{\phi}_i+e_j\sin\omega\boldsymbol{\phi}_j+\omega_2)+O(|s|).$$

It is clear that $K(\hat{d}_2,0,0;\omega_0)=0$. Consequently, the Fréchet derivative of $K(d_2,s,\boldsymbol{W};\omega)$ with respect to $(d_2,s,\boldsymbol{W})$ at $(\hat{d}_2,0,0;\omega_0)$ is given by

$$K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)(d_2,s,\boldsymbol{W})=\boldsymbol{L}(\hat{d}_2)\boldsymbol{W}-d_2e_i\lambda_i\cos\omega_0\begin{pmatrix}0\\\boldsymbol{\phi}_i\end{pmatrix}-d_2e_j\lambda_j\sin\omega_0\begin{pmatrix}0\\\boldsymbol{\phi}_j\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}(-1-\frac{r}{KA_1})\boldsymbol{\phi}_i^2\\\boldsymbol{\phi}_i^2\end{pmatrix}$$
$$+sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}(-1-\frac{r}{KA_2})\boldsymbol{\phi}_i\boldsymbol{\phi}_j\\\boldsymbol{\phi}_i\boldsymbol{\phi}_j\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}(-1-\frac{r}{KA_3})\boldsymbol{\phi}_j^2\\\boldsymbol{\phi}_j^2\end{pmatrix},$$

where A_1,A_2 and A_3 are defined in (17).

Next, we demonstrate that $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0):\mathbb{R}^+\times\mathbb{R}\times X_2\to Y$ is an isomorphism. We decompose

$$\begin{pmatrix}0\\\boldsymbol{\phi}_i\end{pmatrix}=c_1\boldsymbol{\Phi}_i+\begin{pmatrix}S_1\\I_1\end{pmatrix},\quad\begin{pmatrix}0\\\boldsymbol{\phi}_j\end{pmatrix}=c_2\boldsymbol{\Phi}_j+\begin{pmatrix}S_2\\I_2\end{pmatrix},$$

where $\begin{pmatrix}S_1\\I_1\end{pmatrix}=\begin{pmatrix}-c_1\\1-c_1e_i\end{pmatrix}\boldsymbol{\phi}_i$, $c_1=\frac{e_i^*}{1+e_ie_i^*}\neq0$, $\begin{pmatrix}S_2\\I_2\end{pmatrix}=\begin{pmatrix}-c_2\\1-c_2e_j\end{pmatrix}\boldsymbol{\phi}_j$, $c_2=\frac{e_j^*}{1+e_je_j^*}\neq0$. Clearly, $\begin{pmatrix}S_1\\I_1\end{pmatrix}$ and $\begin{pmatrix}S_2\\I_2\end{pmatrix}$ belong to Y_2.

Finally, we proceed by considering the cases $j=2i$ and $i=2j$.

Case 1: $j=2i$.

Simple calculations give $\int_0^\pi\boldsymbol{\phi}_i^2\boldsymbol{\phi}_j\mathrm{d}x=\sqrt{\frac{1}{2\pi}}$, $\int_0^\pi\boldsymbol{\phi}_i\boldsymbol{\phi}_j^2\mathrm{d}x=0$, $\int_0^\pi\boldsymbol{\phi}_j^3\mathrm{d}x=0$, obviously, $\begin{pmatrix}(-1-\frac{r}{KA_3})\boldsymbol{\phi}_j^2\\\boldsymbol{\phi}_j^2\end{pmatrix}\in Y_2$, we decompose

$$\begin{pmatrix}(-1-\frac{r}{KA_1})\boldsymbol{\phi}_i^2\\\boldsymbol{\phi}_i^2\end{pmatrix}=c_3\boldsymbol{\Phi}_j+\begin{pmatrix}S_3\\I_3\end{pmatrix},\quad\begin{pmatrix}(-1-\frac{r}{KA_2})\boldsymbol{\phi}_i\boldsymbol{\phi}_j\\\boldsymbol{\phi}_i\boldsymbol{\phi}_j\end{pmatrix}=c_4\boldsymbol{\Phi}_i+\begin{pmatrix}S_4\\I_4\end{pmatrix},$$

$$c_3=\frac{e_j^*-1-\frac{r}{KA_1}}{1+e_je_j^*}\int_0^\pi\boldsymbol{\phi}_i^2\boldsymbol{\phi}_j\mathrm{d}x=\sqrt{\frac{1}{2\pi}}\frac{e_j^*-1-\frac{r}{KA_1}}{1+e_je_j^*}\neq0,\quad\begin{pmatrix}S_3\\I_3\end{pmatrix}=\begin{pmatrix}-\boldsymbol{\phi}_i^2-\frac{r}{KA_1}\boldsymbol{\phi}_i^2-c_3\boldsymbol{\phi}_j\\\boldsymbol{\phi}_i^2-c_3e_j\boldsymbol{\phi}_j\end{pmatrix}\in Y_2,$$

$$c_4=\frac{e_i^*-1-\frac{r}{KA_2}}{1+e_ie_i^*}\int_0^\pi\boldsymbol{\phi}_i^2\boldsymbol{\phi}_j\mathrm{d}x=\sqrt{\frac{1}{2\pi}}\frac{e_i^*-1-\frac{r}{KA_2}}{1+e_ie_i^*}\neq0,\quad\begin{pmatrix}S_4\\I_4\end{pmatrix}=\begin{pmatrix}-\boldsymbol{\phi}_i\boldsymbol{\phi}_j-\frac{r}{KA_2}\boldsymbol{\phi}_i\boldsymbol{\phi}_j-c_4\boldsymbol{\phi}_i\\\boldsymbol{\phi}_i\boldsymbol{\phi}_j-c_4e_i\boldsymbol{\phi}_i\end{pmatrix}\in Y_2.$$

Right now, we have the following decompositions

$$K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)(d_2,s,\boldsymbol{W})=Z_1+Z_2,$$
$$Z_1=(-d_2c_1e_i\lambda_i\cos\omega_0+sc_4A_2\cos\omega_0\sin\omega_0)\boldsymbol{\Phi}_i+(-d_2c_2e_j\lambda_j\sin\omega_0+sc_3A_1\cos^2\omega_0)\boldsymbol{\Phi}_j\in Y_1,$$
$$Z_2=\boldsymbol{L}(\hat{d}_2)\boldsymbol{W}-d_2e_i\lambda_i\cos\omega_0\begin{pmatrix}S_1\\I_1\end{pmatrix}-d_2e_j\lambda_j\sin\omega_0\begin{pmatrix}S_2\\I_2\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}S_3\\I_3\end{pmatrix}$$
$$+sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}S_4\\I_4\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}(-1-\frac{r}{KA_3})\boldsymbol{\phi}_j^2\\\boldsymbol{\phi}_j^2\end{pmatrix}\in Y_2.$$

Let

$$K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)(d_2,s,\boldsymbol{W})=0. \tag{23}$$

Given that $\boldsymbol{L}(\hat{d}_2)$ is an isomorphism between X_2 and Y_2, (23) equals

$$\begin{cases}(-d_2c_1e_i\lambda_i\cos\omega_0+sc_4A_2\cos\omega_0\sin\omega_0)\boldsymbol{\Phi}_i+(-d_2c_2e_j\lambda_j\sin\omega_0+sc_3A_1\cos^2\omega_0)\boldsymbol{\Phi}_j=0,\\ \boldsymbol{L}(\hat{d}_2)\boldsymbol{W}-d_2e_i\lambda_i\cos\omega_0\begin{pmatrix}S_1\\I_1\end{pmatrix}-d_2e_j\lambda_j\sin\omega_0\begin{pmatrix}S_2\\I_2\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}S_3\\I_3\end{pmatrix}\\ \qquad +sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}S_4\\I_4\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}(-1-\dfrac{r}{KA_3})\boldsymbol{\phi}_j^2\\ \boldsymbol{\phi}_j^2\end{pmatrix}=0.\end{cases}\tag{24}$$

Since ω_0 satisfies (19), the first equation of (24) yields $d_2=0$, $s=0$ and $\boldsymbol{W}=0$ is obtained by substituting them into the second equation, indicating that $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)$ is injective.

Now we prove that $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)$ is surjective. For arbitrary $\begin{pmatrix}S\\I\end{pmatrix}\in Y$, we want to find $(d_2,s,\boldsymbol{W})$ such that

$$K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)(d_2,s,\boldsymbol{W})=\begin{pmatrix}S\\I\end{pmatrix}.\tag{25}$$

By the decomposition of Y, there exist $\alpha,\beta\in\mathbb{R}$ and $\begin{pmatrix}S_0\\I_0\end{pmatrix}\in Y_2$ such that $\begin{pmatrix}S\\I\end{pmatrix}=\begin{pmatrix}S_0\\I_0\end{pmatrix}+\alpha\,\boldsymbol{\Phi}_i+\beta\,\boldsymbol{\Phi}_j$. By substituting the above expression into (25), the resulting equation is obtained

$$\begin{cases}-d_2c_1e_i\lambda_i\cos\omega_0+sc_4A_2\cos\omega_0\sin\omega_0=\alpha,\\ -d_2c_2e_j\lambda_j\sin\omega_0+sc_3A_1\cos^2\omega_0=\beta,\\ \boldsymbol{L}(\hat{d}_2)\boldsymbol{W}-d_2e_i\lambda_i\cos\omega_0\begin{pmatrix}S_1\\I_1\end{pmatrix}-d_2e_j\lambda_j\sin\omega_0\begin{pmatrix}S_2\\I_2\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}S_3\\I_3\end{pmatrix}\\ \qquad +sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}S_4\\I_4\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}(-1-\dfrac{r}{KA_3})\boldsymbol{\phi}_j^2\\ \boldsymbol{\phi}_j^2\end{pmatrix}=\begin{pmatrix}S_0\\I_0\end{pmatrix}.\end{cases}\tag{26}$$

By (19) again, we obtain

$$d_2=\bar{d}_2:=\frac{\alpha c_3A_1\cos\omega_0-\beta c_4A_2\sin\omega_0}{A_2c_2c_4e_j\lambda_j\sin^2\omega_0-A_1c_1c_3e_i\lambda_i\cos^2\omega_0},\ s=\bar{s}:=\frac{\alpha c_2e_j\lambda_j\sin\omega_0-\beta c_1e_i\lambda_i\cos\omega_0}{A_2c_2c_4e_j\lambda_j\cos\omega_0\sin^2\omega_0-A_1c_1c_3e_i\lambda_i\cos^3\omega_0}.$$

By inserting $\bar{d}_2$ and $\bar{s}$ into the third equation of (26), we derive the resulting expression.

$$\begin{aligned}\boldsymbol{L}(\hat{d}_2)\boldsymbol{W}=&\begin{pmatrix}S_0\\I_0\end{pmatrix}+\bar{d}_2e_i\lambda_i\cos\omega_0\begin{pmatrix}S_1\\I_1\end{pmatrix}+\bar{d}_2e_j\lambda_j\sin\omega_0\begin{pmatrix}S_2\\I_2\end{pmatrix}-\bar{s}A_1\cos^2\omega_0\begin{pmatrix}S_3\\I_3\end{pmatrix}\\ &-\bar{s}A_2\cos\omega_0\sin\omega_0\begin{pmatrix}S_4\\I_4\end{pmatrix}-\bar{s}A_3\sin^2\omega_0\begin{pmatrix}(-1-\dfrac{r}{KA_3})\boldsymbol{\phi}_j^2\\ \boldsymbol{\phi}_j^2\end{pmatrix}:=\begin{pmatrix}\tilde{S}\\\tilde{I}\end{pmatrix}\in Y_2.\end{aligned}$$

This implies that $\boldsymbol{W}=\boldsymbol{L}^{-1}\begin{pmatrix}\tilde{S}\\\tilde{I}\end{pmatrix}\in Y_2$. Therefore, $(d_2,s,\boldsymbol{W})=\left(\bar{d}_2,\bar{s},\boldsymbol{L}^{-1}\begin{pmatrix}\tilde{S}\\\tilde{I}\end{pmatrix}\right)$, serves as the solution to (25), demonstrating that $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)$ is surjective.

Thus, $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2,0,0;\omega_0)$ is a bijective map from $\mathbb{R}^+\times\mathbb{R}\times X_2$ to Y. Based on this bijective property, utilizing the implicit function theorem for

$$K(d_2,0,0;\omega_0)=0,\tag{27}$$

we conclude the existence of a curve of nonconstant solutions $(d_2(\omega),s(\omega),\boldsymbol{W}(\omega))$ to (27) (i.e., $\boldsymbol{F}=0$) for sufficiently small $|\omega-\omega_0|$, where ω_0 satisfies (19). Moreover, $(d_2(\omega),s(\omega),\boldsymbol{W}(\omega))$ are continuously differentiable with respect to ω and satisfy the initial conditions $d_2(\omega_0)=\hat{d}_2,s(\omega_0)=0,\boldsymbol{W}(\omega_0)=0$.

Case 2: $i=2j$.

Simple calculation gives $\int_0^\pi\boldsymbol{\phi}_i^2\boldsymbol{\phi}_j\mathrm{d}x=0$, $\int_0^\pi\boldsymbol{\phi}_i\boldsymbol{\phi}_j^2\mathrm{d}x=\sqrt{\dfrac{1}{2\pi}}\neq0$. Then $\begin{pmatrix}(-1-\dfrac{r}{KA_1})\boldsymbol{\phi}_i^2\\ \boldsymbol{\phi}_i^2\end{pmatrix}\in Y_2$ and we decompose

$$\begin{pmatrix}(-1-\dfrac{r}{KA_3})\boldsymbol{\phi}_j^2\\ \boldsymbol{\phi}_j^2\end{pmatrix}=c_5\boldsymbol{\Phi}_i+\begin{pmatrix}S_5\\I_5\end{pmatrix},\quad\begin{pmatrix}(-1-\dfrac{r}{KA_2})\boldsymbol{\phi}_i\boldsymbol{\phi}_j\\ \boldsymbol{\phi}_i\boldsymbol{\phi}_j\end{pmatrix}=c_6\boldsymbol{\Phi}_j+\begin{pmatrix}S_6\\I_6\end{pmatrix},$$

$$\begin{pmatrix} S_5 \\ I_5 \end{pmatrix} = \begin{pmatrix} -\phi_j^2 - \dfrac{r}{KA_3}\phi_j^2 - c_5\phi_i \\ \phi_j^2 - c_5 e_i \phi_i \end{pmatrix},\ c_5 = \frac{e_i^* - 1 - \dfrac{r}{KA_3}}{1 + e_i e_i^*} \int_0^\pi \phi_i \phi_j^2 \mathrm{d}x = \sqrt{\frac{1}{2\pi}} \frac{e_i^* - 1 - \dfrac{r}{KA_3}}{1 + e_i e_i^*},$$

$$\begin{pmatrix} S_6 \\ I_6 \end{pmatrix} = \begin{pmatrix} -\phi_i\phi_j - \dfrac{r}{KA_2}\phi_i\phi_j - c_6\phi_j \\ \phi_i\phi_j - c_6 e_j \phi_j \end{pmatrix},\ c_6 = \frac{e_j^* - 1 - \dfrac{r}{KA_2}}{1 + e_j e_j^*} \int_0^\pi \phi_i \phi_j^2 \mathrm{d}x = \sqrt{\frac{1}{2\pi}} \frac{e_j^* - 1 - \dfrac{r}{KA_2}}{1 + e_j e_j^*}.$$

Hence, we have

$$\begin{aligned} K_{d_2,s,\boldsymbol{W}}(\hat{d}_2, 0, 0; \omega_0)(d_2, s, \boldsymbol{W}) = {} & \boldsymbol{L}(\hat{d}_2)\boldsymbol{W} + (-d_2 c_1 e_i \lambda_i \cos\omega_0 + s c_5 A_3 \sin^2\omega_0)\boldsymbol{\Phi}_i + (-d_2 c_2 e_j \lambda_j \sin\omega_0 + s c_6 A_2 \cos\omega_0 \sin\omega_0)\boldsymbol{\Phi}_j \\ & - d_2 e_i \lambda_i \cos\omega_0 \begin{pmatrix} S_1 \\ I_1 \end{pmatrix} - d_2 e_j \lambda_j \sin\omega_0 \begin{pmatrix} S_2 \\ I_2 \end{pmatrix} + s A_3 \sin^2\omega_0 \begin{pmatrix} S_5 \\ I_5 \end{pmatrix} + s A_2 \cos\omega_0 \sin\omega_0 \begin{pmatrix} S_6 \\ I_6 \end{pmatrix} \\ & + s A_1 \cos^2\omega_0 \begin{pmatrix} (-1 - \dfrac{r}{KA_1})\phi_i^2 \\ \phi_i^2 \end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix} S_1 \\ I_1 \end{pmatrix}$, $\begin{pmatrix} S_2 \\ I_2 \end{pmatrix}$, $\begin{pmatrix} S_5 \\ I_5 \end{pmatrix}$, $\begin{pmatrix} S_6 \\ I_6 \end{pmatrix}$, $\begin{pmatrix} (-1 - \dfrac{r}{KA_1})\phi_i^2 \\ \phi_i^2 \end{pmatrix} \in Y_2$.

As in case 1, it can be demonstrated that if ω_0 satisfies (20), the map $K_{d_2,s,\boldsymbol{W}}(\hat{d}_2, 0, 0; \omega_0)$ is an isomorphism from $\mathbb{R}^+ \times \mathbb{R} \times X_2$ to Y. By the implicit function theorem, the proof is completed.

3.2 Global Bifurcation

In the previous subsection, Theorem 5(i) does not address whether the bifurcation curve Γ_i extends far from the equilibrium point (S^*, I^*). To understand its global structure, a more in-depth analysis is required. This subsection is devoted to examining the global properties of bifurcation solutions arising from simple eigenvalues. Let J denote the closure of the set of nonconstant solutions to (15), and let Γ_i represent the connected component of $J \cup \{(d_{2,i}, (S^*, I^*))\}$.

Theorem 6 Assume the conditions stated in the Theorem 5(i) hold. Then, the projection of the bifurcation curve Γ_i onto the d_2-axis contains the interval $(d_{2,i}, +\infty)$. Furthermore, for any $d_2 > \tilde{d}_2$ and $d_2 \neq d_{2,j}$, where j is a positive integer, system (14) admits at least one nonconstant positive solution.

Proof Rewrite (15) as

$$\begin{cases} -d_1 S'' = a_{11} S + a_{12} I + F^1(S, I), & x \in (0, \pi), \\ -d_2 I'' = a_{21} S + a_{22} I + F^2(S, I), & x \in (0, \pi). \end{cases} \tag{28}$$

The steady state (S^*, I^*) of system (3) is transferred to $(0, 0)$ of system (28), where $F^1(S, I)$ and $F^2(S, I)$ are higher order terms in S and I. Let $G_{d_1} = \left(-d_1 \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + a_{11} \right)^{-1}$, $G_{d_2} = \left(-d_2 \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - a_{22} \right)^{-1}$, then system (28) can be rewritten as $S = G_{d_1}(2a_{11}S) + G_{d_1}(a_{12}I) + G_{d_1}(F^1(S, I))$, $I = G_{d_2}(a_{21}S) + G_{d_2}(F^2(S, I))$. System (14) is then converted into the equation

$$\boldsymbol{U} = K(d_2)\boldsymbol{U} + H(\boldsymbol{U}). \tag{29}$$

Let $K(d_2)\boldsymbol{U} = \left(2a_{11} G_{d_1}(S) + a_{12} G_{d_1}(I),\ a_{21} G_{d_2}(S) \right)$, $H(\boldsymbol{U}) = \left(G_{d_1}(F^1(S, I)),\ G_{d_2}(F^2(S, I)) \right)$. It is noted that $K(d_2)$ is a compact linear operator on X for any fixed $d_2 > 0$. Additionally, $H(\boldsymbol{U})$ satisfies $H(\boldsymbol{U}) = o(|\boldsymbol{U}|)$ as $\boldsymbol{U} \to 0$, uniformly over closed subintervals of d_2 within $(0, \infty)$, and is also a compact operator on X.

To apply Rabinowitz's global bifurcation theorem[20], we first demonstrate that 1 is an eigenvalue of $K(d_{2,i})$ with algebraic multiplicity equal to one. From the Theorem 5(i), it follows that $\ker(K(d_{2,i}) - \boldsymbol{O}) = \ker(\boldsymbol{L}(d_{2,i})) = \mathrm{span}\{\boldsymbol{\Phi}_i\}$. Thus, 1 is indeed an eigenvalue of $K(d_{2,i})$, and $\dim(\ker(K(d_{2,i}) - \boldsymbol{O})) = 1$.

As shown by Jang *et al*[21], the algebraic multiplicity of the eigenvalue 1 of $K(d_{2,i})$ corresponds to the dimension of the generalized null space $\bigcup_{n=1}^{\infty} \ker((K(d_{2,i}) - \boldsymbol{O})^n)$. To establish that the algebraic multiplicity of 1 is one, it suffices to verify equality

$$\ker(K(d_{2,i}) - \boldsymbol{O}) \cap R(K(d_{2,i}) - \boldsymbol{O}) = 0. \tag{30}$$

Let $K^*(d_{2,i})$ denote the adjoint operator of $K(d_{2,i})$. For any $(\boldsymbol{\varphi}, \boldsymbol{\chi}) \in \ker(K^*(d_{2,i}) - \boldsymbol{O})$, the following system holds

$$a_{12} G_{d_1}(\boldsymbol{\varphi}) = \boldsymbol{\chi},\ \ 2a_{11} G_{d_1}(\boldsymbol{\varphi}) + a_{21} G_{d_2}(\boldsymbol{\chi}) = \boldsymbol{\varphi}.$$

By the definitions of G_{d_1} and G_{d_2} the following equations are derived

$$-d_1\chi''=a_{12}\varphi-a_{11}\chi,\ -d_{2,i}\varphi''=f_\varphi\varphi+f_\chi\chi,$$

where $f_\varphi=\dfrac{2a_{11}}{d_1}d_{2,i}+a_{22},\ f_\chi=a_{21}-\dfrac{2a_{11}a_{22}}{a_{12}}-\dfrac{2a_{11}^2}{a_{12}d_1}d_{2,i}$.

Assume

$$\varphi=\sum_{j=0}^{\infty}a_j\phi_j,\ \chi=\sum_{j=0}^{\infty}b_j\phi_j, \tag{31}$$

where ϕ_j are Fourier cosine modes. Substituting into the equations, we obtain

$$\sum_{j=0}^{\infty}\boldsymbol{B}_j^*\begin{pmatrix}a_j\\b_j\end{pmatrix}\phi_j=0,\ \text{where}\ \boldsymbol{B}_j^*=\begin{pmatrix}-d_{2,i}\lambda_j+f_\varphi & f_\chi\\ a_{12} & -d_1\lambda_j-a_{11}\end{pmatrix}.$$

Thus, the kernel of $K^*(d_{2,i})-\boldsymbol{I}$ is given by $\ker(K^*(d_{2,i})-\boldsymbol{O})=\boldsymbol{\Phi}_i',$ where $\boldsymbol{\Phi}_i'=\begin{pmatrix}d_1\lambda_i+a_{11}\\ a_{12}\end{pmatrix}\phi_i$. Here, $\int_0^\pi\boldsymbol{\Phi}_i^{\mathrm{T}}\boldsymbol{\Phi}_i'\mathrm{d}x=2d_1\lambda_i>0,$ and $\boldsymbol{\Phi}_i\notin(\ker(K^*(d_{2,i})-\boldsymbol{O}))^\perp=R(K^*(d_{2,i})-\boldsymbol{O})$. Consequently, the condition (30) is satisfied, and the algebraic multiplicity of the eigenvalue 1 is confirmed to be one.

For $d_2\neq d_{2,i}$ within a small neighborhood of $d_{2,i}$ the linear operator $\boldsymbol{O}-K(d_2):X\to X$ acts as a bijection, and 0 serves as an isolated solution to $(\boldsymbol{O}-K(d_2))\boldsymbol{U}=H(\boldsymbol{U})$. The index of this isolated zero solution is given by

$$i(\boldsymbol{O}-K(d_2)-H,\ (d_2,0))=\deg(\boldsymbol{O}-K(d_2),B,0)=(-1)^p,$$

where B is a sufficiently small ball centered at 0, and p represents the total algebraic multiplicities of eigenvalues of $K(d_2)$ that exceed one.

To demonstrate that the index changes as d_2 crosses $d_{2,i},$ we consider

$$i(\boldsymbol{O}-K(d_{2,i}-\varepsilon)-H,\ (d_{2,i}-\varepsilon,0))\neq i(\boldsymbol{O}-K(d_{2,i}+\varepsilon)-H,\ (d_{2,i}+\varepsilon,0)), \tag{32}$$

where $\varepsilon>0$ is sufficiently small.

Suppose ρ is an eigenvalue of $K(d_2)$ associated with the eigenfunction (φ,χ). Then (φ,χ) satisfies $-\rho d_1\varphi''=(2-\rho)a_{11}\varphi+a_{12}\chi,\ -\rho d_2\chi''=a_{21}\varphi+a_{22}\rho\chi$. Using the Fourier cosine expansion (31), we obtain

$$\sum_{j=0}^{\infty}\begin{pmatrix}(2-\rho)a_{11}-\rho d_1\lambda_j & a_{12}\\ a_{21} & a_{22}\rho-\rho d_2\lambda_j\end{pmatrix}\begin{pmatrix}a_j\\b_j\end{pmatrix}\phi_j=0.$$

This leads to the characteristic equation

$$\rho^2-\frac{2a_{11}}{d_1\lambda_j+a_{11}}\rho-\frac{a_{12}a_{21}}{(d_1\lambda_j+a_{11})(d_2\lambda_j-a_{22})}=0,\ j\geqslant 0. \tag{33}$$

For $d_2=d_{2,i},$ if $\rho=1$ is a root of (33), it follows that $d_{2,i}=d_{2,j},$ implying $i=j$ under the given assumptions. Thus, the eigenvalue associated with $j=i$ is not considered, and the number of eigenvalues of $K(d_2)$ greater than one remains constant near $d_{2,i}$, with their algebraic multiplicities unchanged.

For $j=i$ in (33), let $\rho(d_2)$ and $\tilde{\rho}(d_2)$ denote the roots of the characteristic equation. At $d_2=d_{2,i},$

$$\rho(d_{2,i})=1,\ \tilde{\rho}(d_{2,i})=\frac{a_{11}-d_1\lambda_j}{a_{11}+d_1\lambda_j}<1.$$

For d_2 close to $d_{2,i},$ we have $\tilde{\rho}(d_2)<1$. Since $\rho(d_2)$ is an increasing function of $d_2,$ it follows that

$$\rho(d_{2,i}+\varepsilon)>1,\ \rho(d_{2,i}-\varepsilon)<1.$$

Consequently, $K(d_{2,i}+\varepsilon)$ possesses exactly one additional eigenvalue greater than 1 compared to $K(d_{2,i}-\varepsilon)$. Furthermore, by applying a similar reasoning as above, we can demonstrate that this eigenvalue has an algebraic multiplicity of one. Thus, equation (32) holds. According to Ref. [20], it can be deduced that Γ_i must satisfy one of the following conditions: (i) Γ_i meets infinity in $\mathbb{R}\times X,$ or (ii) Γ_i meets $(d_{2,k},\ (S^*,I^*))$ for some $k\neq i$ with $d_{2,k}>0$.

In the following, we utilize the methods from Nishiura[22] and Takagi[23] to establish that the bifurcation curve Γ_i must extend indefinitely in $\mathbb{R}\times X$. For further insights, detailed reasoning can also be found in Jang *et al*[21]. For the sake of contradiction, assume that Γ_i lies in a bounded region. This implies that Γ_i is compact and intersects additional bifurcation points. Let k be an integer such that Γ_i includes $(d_{2,k},\ (S^*,I^*)),$ but does not include $(d_{2,j},\ (S^*,I^*))$ for any $j>k$. Consider system (14) over the interval $(0,\ \pi/k)$.

Suppose that $\tilde{\boldsymbol{U}}$ is a solution to (14) over the interval (0, π/k). By applying a reflective and periodic extension as described by Nishiura[22] and Takagi[23], a corresponding solution $\bar{\boldsymbol{U}}$ to (14) can be constructed. Let $x_n=n\pi/k,$ where $n=$

$0, 1, 2, \cdots, k,$ and define

$$\bar{\boldsymbol{U}}=\begin{cases}\tilde{\boldsymbol{U}}(x-x_{2n}), & \text{if } x_{2n}\leqslant x\leqslant x_{2n+1},\\ \tilde{\boldsymbol{U}}(x_{2n+2}-x), & \text{if } x_{2n+1}\leqslant x\leqslant x_{2n+2}.\end{cases}$$

Let $\tilde{d}_{2,i}=\dfrac{a_{11}a_{22}-a_{12}a_{21}-a_{22}d_1\tilde{\lambda}_i}{\tilde{\lambda}_i(a_{11}-d_1\tilde{\lambda}_i)}$. It is well known that the operator $u\mapsto -u_{xx}$ under no flux boundary conditions on the interval $(0, \pi/k)$ has eigenvalues $\tilde{\lambda}_i$, where $\tilde{\lambda}_i=(ik)^2=\lambda_{ki}$. Consequently, it follows that $\tilde{d}_{2,i}=d_{2,ki}$ for $i=0, 1, 2, \cdots,$ and specifically, $\tilde{d}_{2,1}=d_{2,k}$. Using the argument presented in Theorem 5, it can be deduced that $(\tilde{d}_{2,1}, (S^*, I^*))=(d_{2,k}, (S^*, I^*))$ also constitutes a bifurcation point for system (14) over the interval $(0, \pi/k)$. Let Λ_k denote the solution branch that bifurcates from $(\tilde{d}_{2,1}, (S^*, I^*))$.

Following a similar line of reasoning, it can be demonstrated that Λ_k either extends toward infinity or intersects a point of the form $(\tilde{d}_{2,s}, (S^*, I^*))=(\tilde{d}_{2,ks}, (S^*, I^*))$ for some $s>1$. If the latter case was true, it would lead to a contradiction. Specifically, it would imply that Γ_i intersects $(\tilde{d}_{2,sk}, (S^*, I^*))$ for some $s>1,$ which is inconsistent with the definition of k. Therefore, we conclude that Λ_k must extend to infinity. By the same reasoning, Γ_i is also unbounded. The proof is completed.

3.3 Direction of the Bifurcation Solutions

In this subsection, we analyze the direction of the steady-state bifurcation corresponding to the simple eigenvalues identified in Theorem 5(i). Rewrite the map $\boldsymbol{F}$: $\mathbb{R}^+\times X\to Y$ by

$$\boldsymbol{F}(d_2, \boldsymbol{U})=\begin{pmatrix}d_1S''+\tilde{f}(S, I)\\ d_2I''+\tilde{g}(S, I)\end{pmatrix},$$

where

$$\tilde{f}(S, I)=r(S+S^*)\left(1-\frac{S+S^*}{k}\right)-\frac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-\mu(S+S^*)+c(I+I^*),$$

$$\tilde{g}(S, I)=\frac{\beta(S+S^*)(I+I^*)}{a(S+S^*)+b(I+I^*)}-(\mu+\tau+c)(I+I^*)-(I+I^*)v.$$

Then a straightforward calculation gives

$$\begin{cases}\tilde{f}_S(0,0)=r-\dfrac{2rS^*}{k}-\dfrac{b\beta(I^*)^2}{(aS^*+bI^*)^2}-\mu,\ \tilde{f}_I(0,0)=-\dfrac{a\beta(S^*)^2}{(aS^*+bI^*)^2}+c,\ \tilde{f}_{SS}(0,0)=-\dfrac{2r}{k}+\dfrac{2ab\beta(I^*)^2}{(aS^*+bI^*)^3},\\ \tilde{f}_{SI}(0,0)=-\dfrac{2ab\beta S^*I^*}{(aS^*+bI^*)^3},\ \tilde{f}_{II}(0,0)=\dfrac{2ab(aS^*+bI^*)\beta(S^*)^2}{(aS^*+bI^*)^4},\ \tilde{f}_{SSI}(0,0)=\dfrac{2ab^2\beta(I^*)^2-4a^2b\beta S^*I^*}{(aS^*+bI^*)^4},\\ \tilde{f}_{SSS}(0,0)=\dfrac{-6a^2b\beta(I^*)^2}{(aS^*+bI^*)^4},\ \tilde{f}_{SII}(0,0)=\dfrac{4ab^2\beta S^*I^*-2a^2b\beta(S^*)^2}{(aS^*+bI^*)^4},\ \tilde{f}_{III}(0,0)=\dfrac{-6ab^2\beta(S^*)^2}{(aS^*+bI^*)^4},\end{cases}$$

$$\begin{cases}\tilde{g}_S(0,0)=\dfrac{b\beta(I^*)^2}{(aS^*+bI^*)^2},\ \tilde{g}_I(0,0)=\dfrac{a\beta(S^*)^2}{(aS^*+bI^*)^2}-(\mu+\tau+c+v),\ \tilde{g}_{SS}(0,0)=\dfrac{-2ab\beta(I^*)^2}{(aS^*+bI^*)^3},\ \tilde{g}_{SI}(0,0)=\dfrac{2ab\beta S^*I^*}{(aS^*+bI^*)^3},\\ \tilde{g}_{II}(0,0)=\dfrac{-2ab\beta(S^*)^2}{(aS^*+bI^*)^3},\ \tilde{g}_{SSI}(0,0)=\dfrac{2ab^2\beta(I^*)^2-4a^2b\beta S^*I^*}{(aS^*+bI^*)^4},\ \tilde{g}_{SSS}(0,0)=\dfrac{6a^2b\beta(I^*)^2}{(aS^*+bI^*)^4},\\ \tilde{g}_{SII}(0,0)=\dfrac{2a^2b\beta(S^*)^2-4ab^2\beta S^*I^*}{(aS^*+bI^*)^4},\ \tilde{g}_{III}(0,0)=\dfrac{6ab^2\beta(S^*)^2}{(aS^*+bI^*)^4}.\end{cases}$$

According to Theorem 5(i), we have $\dim\ker \boldsymbol{F}_U(d_{2,i},(0,0))=\operatorname{codim} R(\boldsymbol{F}_U(d_{2,i},(0,0)))=1,$ and $\ker \boldsymbol{F}_U(d_{2,i}, (0,0))=\operatorname{span}\{\boldsymbol{\Phi}_i\}$.

Consequently, the spaces X and Y can be decomposed as follows

$$X=\ker \boldsymbol{F}_U(d_{2,i}, (0,0))\oplus Z,\ \ Y=R(\boldsymbol{F}_U(d_{2,i}, (0,0)))\oplus Z',$$

where Z and Z' are the respective complements of $\ker \boldsymbol{F}_U(d_{2,i}, (0,0))$ in X and $R\left(\boldsymbol{F}_U\left(d_{2,i},(0,0)\right)\right)$ in Y. Using (21), we derive $\left\langle \boldsymbol{F}_{d_2U}(d_{2,i}, (0,0))\boldsymbol{\Phi}_i, \boldsymbol{\Phi}_i^*\right\rangle=-\lambda_i e_i e_i^*=-i^2e_ie_i^*\neq 0.$

According to Ref. [24], we obtain that

$$d_2'(0)=-\frac{\left\langle \boldsymbol{F}_{UU}(d_{2,i}, (0,0))\boldsymbol{\Phi}_i^2, \boldsymbol{\Phi}_i^*\right\rangle}{2\left\langle \boldsymbol{F}_{d_2U}(d_{2,i}, (0,0))\boldsymbol{\Phi}_i, \boldsymbol{\Phi}_i^*\right\rangle}.$$

By quick computation, we get $\left\langle \boldsymbol{F}_{UU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^2,\ \boldsymbol{\Phi}_i^* \right\rangle = (o_i + p_i e_i^*) \int_0^\pi \boldsymbol{\phi}_i^3 \mathrm{d}x,$ where

$$o_i = \tilde{f}_{SS}(0,0) + 2\tilde{f}_{SI}(0,0)e_i + \tilde{f}_{II}(0,0)e_i^2,\ \ p_i = \tilde{g}_{SS}(0,0) + 2\tilde{g}_{SI}(0,0)e_i + \tilde{g}_{II}(0,0)e_i^2.$$

Since $\boldsymbol{\phi}_i = \sqrt{\frac{2}{\pi}} \cos ix,$ we have $\int_0^\pi \boldsymbol{\phi}_i^3 \mathrm{d}x = 0,$ this results in $< \boldsymbol{F}_{UU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^2,\ \boldsymbol{\Phi}_i^* >= 0$. Thus, $d_2'(0) = 0$.

At the same time, Shi[24] indicated that the steady state bifurcation was supercritical (or subcritical) in direction if

$$d_2''(0) = -\frac{\left\langle \boldsymbol{F}_{UUU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^3, \boldsymbol{\Phi}_i^* \right\rangle + 3\left\langle \boldsymbol{F}_{UU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i\boldsymbol{\theta},\ \boldsymbol{\Phi}_i^* \right\rangle}{3\left\langle \boldsymbol{F}_{d_2 U}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^2,\ \boldsymbol{\Phi}_i^* \right\rangle} > (<) 0,$$

where $\boldsymbol{\theta}$ is the solution of the following problem

$$\boldsymbol{F}_{UU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^2 + \boldsymbol{F}_U(d_{2,i},\ (0,0))\boldsymbol{\theta} = 0.$$

By direct calculation, we obtain

$$\left\langle \boldsymbol{F}_{UUU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i^3,\ \boldsymbol{\Phi}_i^* \right\rangle = \frac{4}{\pi^2}(m_i + n_i e_i^*) \int_0^\pi \cos^4\ (ix)\mathrm{d}x = \frac{3}{2\pi}(m_i + n_i e_i^*),$$

where $m_i = \tilde{f}_{SSS}(0,0) + 3\tilde{f}_{SSI}(0,0)e_i + 3\tilde{f}_{SII}(0,0)e_i^2 + \tilde{f}_{III}(0,0)e_i^3,\ n_i = \tilde{g}_{SSS}(0,0) + 3\tilde{g}_{SSI}(0,0)e_i + 3\tilde{g}_{SII}(0,0)e_i^2 + \tilde{g}_{III}(0,0)e_i^3$.

Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2),$ and it satisfies

$$\begin{cases} d_1\boldsymbol{\theta}_1'' + \tilde{f}_S(0,0)\boldsymbol{\theta}_1 + \tilde{f}_I(0,0)\boldsymbol{\theta}_2 = -o_i\boldsymbol{\phi}_i^2, & x \in (0,\pi), \\ d_{2,i}\boldsymbol{\theta}_2'' + \tilde{g}_S(0,0)\boldsymbol{\theta}_1 + \tilde{g}_I(0,0)\boldsymbol{\theta}_2 = -p_i\boldsymbol{\phi}_i^2, & x \in (0,\pi), \\ \boldsymbol{\theta}_i'(0) = \boldsymbol{\theta}_i'(\pi) = 0, & i = 1,2. \end{cases} \tag{34}$$

Integrating (34) by part, we derive

$$\int_0^\pi \boldsymbol{\theta}_1 \mathrm{d}x = \frac{p_i\tilde{f}_I(0,0) - o_i\tilde{g}_I(0,0)}{\tilde{f}_S(0,0)\tilde{g}_I(0,0) - \tilde{f}_I(0,0)\tilde{g}_S(0,0)} := \gamma_1,\ \ \int_0^\pi \boldsymbol{\theta}_2 \mathrm{d}x = \frac{o_i\tilde{g}_S(0,0) - p_i\tilde{f}_S(0,0)}{\tilde{f}_S(0,0)\tilde{g}_I(0,0) - \tilde{f}_I(0,0)\tilde{g}_S(0,0)} := \gamma_2. \tag{35}$$

In addition, a simple calculation yields $\left\langle \boldsymbol{F}_{UU}(d_{2,i},\ (0,0))\boldsymbol{\Phi}_i\boldsymbol{\theta},\ \boldsymbol{\Phi}_i^* \right\rangle = C_1\int_0^\pi \boldsymbol{\theta}_1\boldsymbol{\phi}_i^2 \mathrm{d}x + C_2\int_0^\pi \boldsymbol{\theta}_2\boldsymbol{\phi}_i^2 \mathrm{d}x,$ where

$$C_1 = \tilde{f}_{SS}(0,0) + \tilde{f}_{SI}(0,0)e_i + (\tilde{g}_{SS}(0,0) + \tilde{g}_{SI}(0,0)e_i)e_i^*,\ C_2 = \tilde{f}_{SI}(0,0) + \tilde{f}_{II}(0,0)e_i + (\tilde{g}_{SI}(0,0) + \tilde{g}_{II}(0,0)e_i)e_i^*.$$

In order to finish our computations later, we now compute $\int_0^\pi \boldsymbol{\theta}_1\boldsymbol{\phi}_i^2 \mathrm{d}x,$ and $\int_0^\pi \boldsymbol{\theta}_2\boldsymbol{\phi}_i^2 \mathrm{d}x$. Multiplying (34) by $\boldsymbol{\phi}_i^2$ and integrating, we get $\int_0^\pi \boldsymbol{\phi}_i^4 \mathrm{d}x = \frac{4}{\pi^2}\int_0^\pi \cos^4\ (ix)\mathrm{d}x = \frac{3}{2\pi},$ then

$$d_1\int_0^\pi \boldsymbol{\theta}_1''\boldsymbol{\phi}_i^2 \mathrm{d}x + \tilde{f}_S(0,0)\int_0^\pi \boldsymbol{\theta}_1\boldsymbol{\phi}_i^2 \mathrm{d}x + \tilde{f}_I(0,0)\int_0^\pi \boldsymbol{\theta}_2\boldsymbol{\phi}_i^2 \mathrm{d}x = -\frac{3}{2\pi}o_i, \tag{36}$$

$$d_{2,i}\int_0^\pi \boldsymbol{\theta}_2''\boldsymbol{\phi}_i^2 \mathrm{d}x + \tilde{g}_S(0,0)\int_0^\pi \boldsymbol{\theta}_1\boldsymbol{\phi}_i^2 \mathrm{d}x + \tilde{g}_I(0,0)\int_0^\pi \boldsymbol{\theta}_2\boldsymbol{\phi}_i^2 \mathrm{d}x = -\frac{3}{2\pi}p_i, \tag{37}$$

where $\int_0^\pi \boldsymbol{\theta}_j''\boldsymbol{\phi}_i^2 \mathrm{d}x = \frac{4i^2}{\pi}\int_0^\pi \boldsymbol{\theta}_j\ (1 - \pi\boldsymbol{\phi}_i^2)\mathrm{d}x,\ \ j = 1,2.$

Substituting (35) into (36) and (37), we have

$$\int_0^\pi \boldsymbol{\theta}_1\boldsymbol{\phi}_i^2 \mathrm{d}x = \frac{\delta_1(\tilde{g}_I(0,0) - 4i^2 d_{2,i}) - \delta_2\tilde{f}_I(0,0)}{(\tilde{f}_S(0,0) - 4i^2 d_1)(\tilde{g}_I(0,0) - 4i^2 d_{2,i}) - \tilde{g}_S(0,0)\tilde{f}_I(0,0)} := L_1,$$

$$\int_0^\pi \boldsymbol{\theta}_2\boldsymbol{\phi}_i^2 \mathrm{d}x = \frac{\delta_1\tilde{g}_S(0,0) - \delta_2(\tilde{f}_S(0,0) - 4i^2 d_1)}{\tilde{f}_I(0,0)\tilde{g}_S(0,0) - (\tilde{g}_I(0,0) - 4i^2 d_{2,i})(\tilde{f}_S(0,0) - 4i^2 d_1)} := L_2,$$

where $\delta_1 = -\frac{3o_i}{2\pi} - \frac{4i^2 d_1}{\pi}\gamma_1,\ \delta_2 = -\frac{3p_i}{2\pi} - \frac{4i^2 d_{2,i}}{\pi}\gamma_2$. In total, we obtain

$$d_2''(0) = \frac{(m_i + n_i e_i^*) + 2\pi(C_1 L_1 + C_2 L_2)}{2\pi e_i e_i^* \lambda_i}. \tag{38}$$

Therefore, we may construct the following theorem to ascertain the direction of the local bifurcation by figuring out the sign of $d_2''(0)$, and $\lambda_i = i^2$.

Theorem 7 For the bifurcation point $(\hat{d}_2,\ (0,0))$ identified in Theorem 5(i), the bifurcation is classified as supercritical if $d_2''(0) > 0$ and subcritical if $d_2''(0) < 0$. Here, i belongs to the interval $[1,\ i_0]$, where i_0 denotes the largest positive integer satisfying $d_1\lambda_i < a_{11}$.

4 Numerical Analysis and Simulations

In this section, on the basis of the foregoing analysis, corresponding numerical simulations are presented to verify the foregoing conclusions. First, we numerically solve the initial boundary value problem for system (3) and system (4) on $\Omega=(0,\pi)$ via a spatially discrete finite difference method.

In Fig. 1, the equilibrium point (S^*, I^*) of system (4) is locally asymptotically stable and converges to the positive constant solution (0.31, 0.24) over time for the following parameter values: $a=1$, $\beta=2.4$, $b=1$, $v=0.1$, $c=0.04$, $\mu=0.01$, $K=1$, $\tau=1.2$, and $r=1.5$.

The neutral curve of d_2 as a function of $i\in\mathbb{R}$ can be seen in Fig. 2. Obviously, when $d_{2,1}\neq d_{2,2}$ lies on the left side of Fig. 2, it means that the system has a steady-state bifurcation at simple eigenvalues; when $d_{2,1}=d_{2,2}$ lies on the right side of Fig. 2, it means that the system has a steady-state bifurcation at double eigenvalues.

The diffusion coefficient d_2 is selected as the control parameter to investigate its effect on the model dynamics via a finite-difference scheme, with the spatial region taken to be $[0, 100]\times[0, 100]$, and the boundary condition taken to be the Neumann boundary condition. For convenience, we set $d_1=0.03$. According to Theorem 3, the equilibrium point E^* of the system of partial differential equations (2) is Turing unstable if $d_2>\tilde{d}_2$ and system (3) will exhibit Turing patterns. In the next numerical simulation, the concentration values of $S(t,\cdot)$ and the infected population $I(t,\cdot)$ are expressed as a ratio from red (maximum) to blue (minimum). From the numerical simulation, the Turing region can be derived.

In Fig. 3, the steady-state Turing patterns for the susceptible population $S(t,\cdot)$ and the infected population $I(t,\cdot)$ are shown for $d_2=1.2$. It can be seen that, starting from the homogeneous state, the concentration values in panels (A1) and (A2) are essentially unchanged, with the initial values chosen as the equilibrium solution plus a stochastic perturbation; bar-like patches appear in panels (B1) and (B2); the stochastic perturbation then leads to a decay of the voids, and bar-like patches appear in panels (C1) and (C2); eventually, bars in panels (D1) and (D2) occupy almost the entire region, and the dynamical behaviour of the system no longer changes.

Figure 4 illustrates the temporal evolution of the spatial patterns of the susceptible population $S(t,\cdot)$ in both two-

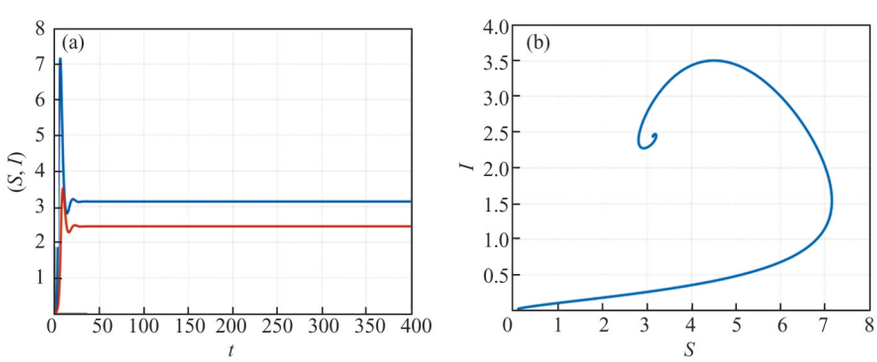


Fig. 1 Time diagrams (a) and phase diagrams (b)

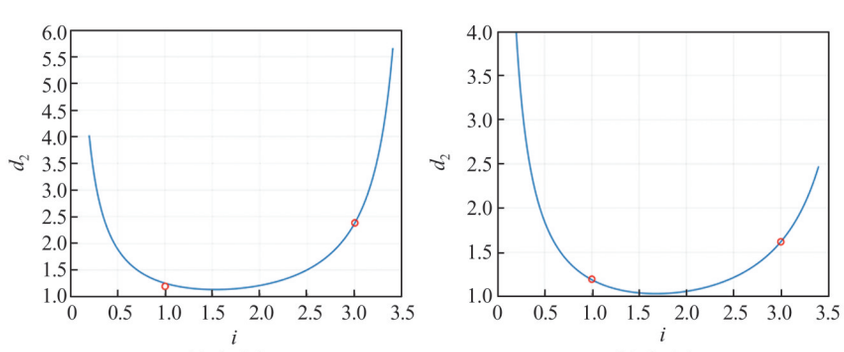


Fig. 2 The neutral curves d_2 about $i\in\mathbb{R}$ for $\tau=1.2$, $a=1$, $\beta=2.4$, $b=1$, $v=0.1$, $c=0.04$, $\mu=0.01$, $K=1$, $r=1.5$

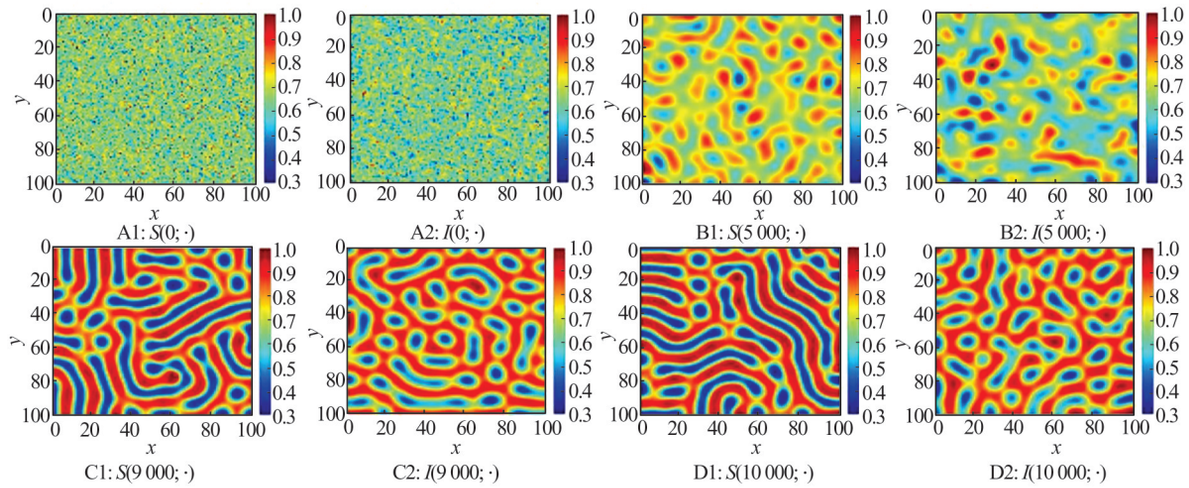


Fig. 3 Pattern formation in $S(t;\cdot)$ and $I(t;\cdot)$ for system (2)

and three-dimensional representations for $d_2 = 1.5$. Initially, the population is distributed almost homogeneously with small random perturbations, as shown in panels (E1) and (E2). At $t = 2\,000$, these perturbations are amplified, and isolated spot-like structures begin to emerge in panels (F1) and (F2). As the system evolves to $t = 6\,000$, the bars become more pronounced and gradually connect to form a mixed pattern composing of bars and short stripe-like structures in panels (G1) and (G2). At $t = 8\,000$, a well-developed and persistent spatial pattern forms, characterized by the coexistence of densely distributed spots and irregular stripes in panels (H1) and (H2). The corresponding two- and three-dimensional plots show consistent spatial distributions, while the three-dimensional representations further reveal the variation in population density over the spatial domain.

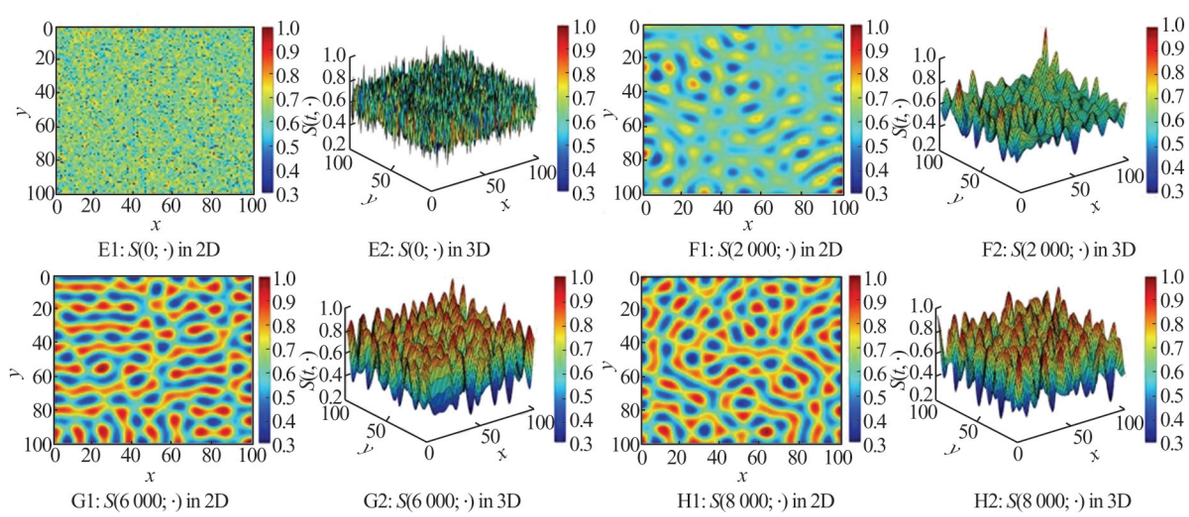


Fig. 4 Pattern formation in $S(t;\cdot)$ and $I(t;\cdot)$ for system (2) in 2D and 3D space

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一类具有扩散项的SIR传染病模型的分歧与斑图动力学分析

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摘要：本文研究一类带扩散的易感-感染-康复（SIR）传染病模型。该模型引入心理影响系数a和b，采用饱和发生率刻画疾病带来的心理效应。首先，推导正解的先验估计，利用能量方法证明非常数正解的不存在性。其次，分析正平衡点的图灵失稳，并借助Leray-Schauder度理论证明非常数正解的存在性。随后，利用分支理论刻画稳态分支的局部结构与全局结构。通过空间分解和隐函数定理，构造二重特征值处稳态分支的局部结构并推导出判定分支方向的判别准则。最后，数值模拟对理论结果进行补充与验证。

关键词：分歧；斑图；SIR模型；数值模拟

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